

Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

Lecture 1 - Jan 8th 2024

Chapter 1 Iteration and Orbits

allows for indefinite iteration

Defn Let $f: A \rightarrow \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$ [I.E. $f: A \rightarrow A$]. For $a \in A$, we may iterate the function at a :

$$x_1 = a$$

$$x_2 = f(a)$$

$$x_3 = f(f(a)) = f^2(a)$$

$$x_4 = f(f(f(a))) = f^3(a)$$

$\vdots$

all will exist in A .

We call $(x_n)_{n=1}^{\infty} = (x_n)$ we call that sequence the orbit of a under f .

Ex. $f(x) = x^4 + 2x^2 - 2$, $a = -1$

$x_1 = a, x_2 = f(a), x_3 = f^2(a) \dots$

$-1, 1, 1, \dots \rightarrow$ eventually constant / periodic

Ex. $f(x) = -x^2 - x + 1$, $a = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \dots$

$0, 1, -1, 1, -1, 1, \dots \rightarrow$ eventually periodic w period 2

Ex. $f(x) = x^2 - 3x + 1$, $a = 1$

$x_1, x_2, x_3, x_4, \dots$

$1, -1, 3, 19 \rightarrow \infty$ goes to infinity [he will talk about convergence / divergence later on]

Ex. $f(x) = x^2 + 2x$, $a = -0.5$

My Q: Does it ever equal 1, is that a distinction?

x_1, x_2, x_3, x_4

$-0.5, -0.75, -0.9375, -0.9961, \dots$ converges to -1

Ex. $f(x) = x^3 - 3x$, $a = 0.75$

$x_1, x_2, x_3, x_4, x_5, \dots$

$0.75, -1.828, -0.625, 1.631, -0.552, \dots$

This is chaotic behaviour.

* For PMATH insights: an interval around zero is dense *

Defn: $f: A \rightarrow \mathbb{R}$, $f(A) \subseteq A$. We say $a \in A$ is a fixed point in f iff $f(a) = a$. In this case, the orbit of a is $a, a, a, \dots$ which is constant.

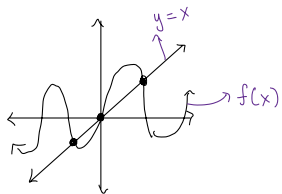
Ex. Find all fixed points of $f(x) = x^2 + x - 4$

When does $x^2 + x - 4 = x$?

$$\Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2$$

Will often give low degree polynomial

Ex. How many fixed points does $f(x)$ as pictured below have?



Geometrically, a fixed point occurs when $f(x)$ intersects $y = x$

Ex. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

$$\text{Solve for } x^4 - 3x + 1 = x$$

$$\Rightarrow x^4 - 4x + 1 = 0$$

Intermediate Value theorem



Since $g(x)$ is continuous $g(0) = 1 > 0$ and $g(1) = -2 < 0$. By IVT $\exists x \in (0, 1)$ s.t. $g(x) = 0 \Leftrightarrow f(x) = x$.

Defn $f: A \rightarrow \mathbb{R}$, $f(A) \subseteq A$

① We say $a \in A$ is a **periodic point** for f if its orbit is periodic. I.E. $\exists n \in \mathbb{N}$ $f^n(a) = a$. The least such n is called the period of a and/or the orbit.

② **Eventually periodic** $\exists n < m$, $f^n(a) = f^m(a)$

Lecture 2 - Jan 10th 2024

Def'n [Doubling Function]

$D: [0, 1) \rightarrow [0, 1)$ where $D(x) = \text{fractional part of } 2x$ aka $2x \bmod 1$

Ex) $D(0.4) = 0.8$

$$D(0.6) = 0.2$$

$$D(0.8) = 0.6$$

$$D(0.5) = 0$$

It is an important function as it also provides a rich source of periodic orbits

Ex) $D, a = \frac{1}{5}$

Orbit: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \dots$ This has period 4. This leads into some cute number theory with GCDs

Ex) $D, a = \frac{1}{20}$

Orbit $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \dots$ This is eventually periodic with period 4.

The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given f and a , does $f^n(a)$ tend towards some limit L ? This does happen surprisingly often

The language of this course vs Elementary Real Analysis

Notation:

If $(x_n)_{n=1}^{\infty}$ is a sequence of real number we write $(x_n) \subseteq \mathbb{R}$ $\rightarrow$ a small abuse of notation but reasonable

Def'n $(x_n) \subseteq \mathbb{R}, x \in \mathbb{R}$ we say (x_n) converges to x iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ for all $n \geq N$

$\downarrow$
* N depends *
on ϵ

We write $x_n \rightarrow x$ or $\lim x_n = x$

Ex) Claim: $\frac{1}{n} \rightarrow 0$, Let $\epsilon > 0$ be given.

Note: $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Consider $N = \frac{2}{\epsilon}$ For $n \geq N$ we have $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$
 $\rightarrow$ not necessarily a Natural number. Can always use ceiling function $\lceil \cdot \rceil$ to bring this to a natural number
 $\hookrightarrow$ bring this to $2/\epsilon$ so that we ensure strictly less than epsilon.

Ex) claim $\frac{2n}{n+3} \rightarrow 2$

Let $\epsilon > 0$ be given. Let us choose $N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \frac{\epsilon}{6}$$

$$\begin{aligned} \text{For } n \geq N \quad & \left| \frac{2n}{n+3} - 2 \right| \\ &= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right| \\ &= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n} \quad \text{if you divide by less of something you get something bigger} \\ &\frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\epsilon}{6}\right) = \epsilon \end{aligned}$$

Defn $(x_n) \subseteq \mathbb{R}$ we say (x_n) is **bounded** if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop $(x_n) \subseteq \mathbb{R}$ If (x_n) is convergent, then (x_n) is bounded.

Ex) $x_n = (-1)^n$ Shows converse is not true

Proof Suppose $x_n \rightarrow x$. then $\exists N \in \mathbb{N}, n \geq N \Rightarrow |x_n - x| < 1$ ↗ picked ϵ , Blake's favourite is 1

For $n \geq N$, ↗ this is the reverse triangle inequality, make sure you know how to prove from triangle inequality
 $|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x|$ This is only true for $n \geq N$

So let $M = \max \{ |x_1|, \dots, |x_{N-1}|, 1 + |x| \}$ ■

Prop let $x_n \rightarrow x, y_n \rightarrow y$

Not super important to us,
value is in working with
defn of convergence

$$(1) \quad x_n + y_n \rightarrow x + y$$

$$(2) \quad x_n y_n \rightarrow xy$$

Proof:

(1) Let $\epsilon > 0$ be given. There exists $N_1, N_2 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$$

For $N = \max \{N_1, N_2\}$ and $n \geq N$

$$|x_n + y_n - (x + y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

② Let $\varepsilon > 0$ be given

Note •

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \quad \star$$

Since (x_n) is bounded $\exists M > 0, \forall n, |x_n| < M$

Let $N_1, N_2 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$ ↪ make sure not dividing by zero

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$$

For $n \geq N := \max \{N_1, N_2\}$ we have $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $\star$ ■

Lec 3 - Jan 12th 2024

Def'n we say $(X_n) \subseteq \mathbb{R}$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |x_n - x_m| < \epsilon$

Point of Clarification. The Wednesday class before the assignment is due we will cover all the material needed for the assignment. Also we can use Calculus 2. A lot of this course is in the language of Calculus and Real Analysis

Prop Convergent $\Rightarrow$ Cauchy

Proof: Let $\epsilon > 0$ be given and suppose (X_n) is convergent. Say $x_n \rightarrow x \in \mathbb{R}$. There exist $N \in \mathbb{N}$ $n \geq N$

$\Rightarrow |x_n - x| < \boxed{\frac{\epsilon}{2}}$ Then for $n, m \geq N$

$\hookrightarrow$ both are $< \epsilon$ which gives information on what should be in the box

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

$\hookrightarrow$ take ϵ divided by # of parts

□

Fact: $(X_n) \subseteq \mathbb{R}, (X_n)$ Cauchy $\Leftrightarrow (X_n)$ Convergent. This is also apart of the Completeness of $\mathbb{R}$

Big Idea: To prove (X_n) is Cauchy you do not have to guess the limit! $\hookrightarrow$ this will be useful for fixed points

Def'n $f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. we say f is **continuous** at a iff $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $x \in A$ and $|x - a| < \delta$.

$\hookrightarrow$ close values in the domain gives close values in function

Fact: f is continuous at a iff $\forall (x_n) \subseteq A$ with $x_n \rightarrow a$, we have $f(x_n) \rightarrow f(a)$. Provides link between convergence and continuity.

$\hookrightarrow$ can prove as an exercise

$\hookrightarrow$ highly dependent on that f goes from $[a,b] \rightarrow [a,b]$

Prop: If $f: [a,b] \rightarrow [a,b]$ is continuous then $f(x)$ has a fixed point

Proof: we know $f(a) \geq a$ and $f(b) \leq b$. $\Leftrightarrow f(a) - a \geq 0$ and $f(b) - b \leq 0$

By applying the IVT to the continuous function $g(x) = f(x) - x$ $\exists x \in [a,b]$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ □

$\hookrightarrow$ a preview of why Calculus is applicable to this course. Epsilon-delta will come back.

Def'n $f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}$. we say that f is a **contraction** iff $\exists C \in [0,1)$ s.t. $\forall x, y \in A, |f(x) - f(y)| \leq C|x - y|$

$\hookrightarrow$ is a type of Lipschitz fcn $\hookrightarrow$ would that mean fcn. is below line $x=y$ graphically?

Prop Contractions are continuous at every point

Proof Let $\epsilon > 0$ be given. Suppose $|f(x) - f(y)| \leq C|x - y|$ as before. Fix $y \in A$, Consider $\delta = \boxed{\frac{\epsilon}{C+1}}$ $\hookrightarrow$ to make sure we aren't dividing by 0

and Assume $x \in A$ s.t. $|x-y| < \delta$. Then

$$|f(x) - f(y)| \leq C|x-y| < C\delta < \epsilon \quad \square$$

Sqrt function a problematic, not a contraction bc it drops off at 0

Defn We say $A \subseteq \mathbb{R}$ is **close** iff whenever $(x_n) \subseteq A$ with $x_n \rightarrow x \in \mathbb{R}$ then $x \in A$.

Ex. $[a, b]$ is closed

Ex. $(0, 1]$ not closed $\frac{1}{n} \rightarrow 0 \notin (0, 1]$

Theorem [Banach Contraction Mapping Theorem] Suppose $A \subseteq \mathbb{R}$ is closed and $f: A \rightarrow A$ is a contraction.

Then there exist a unique fixed point $a \in A$ for f . Moreover $\forall x \in A \quad f^n(x) \rightarrow a$ ↪ has a cute proof. allegedly. will prove on Monday

Ex $f: [0, 1] \rightarrow [0, 1] \quad f(x) = \frac{1}{3-x}$

Note: $\frac{1}{3} \leq f(x) \leq \frac{1}{2} \rightarrow$ shows it maps back into $[0, 1]$
↪ also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \quad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT, $\forall x, y \in [0, 1] \quad \exists c \in (0, 1) \quad \text{s.t.} \quad f(x) - f(y) = f'(c)(x-y)$

$$\Rightarrow |f(x) - f(y)| = |f'(c)| |x-y| \leq \underbrace{\frac{1}{4}}_C |x-y|$$

$\therefore f(x)$ is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow 1 = 3x - x^2$$

$$\Leftrightarrow x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow x = \frac{3-\sqrt{5}}{2}$$

$$\therefore \forall x \in [0, 1] \quad f^n(x) \rightarrow \frac{3-\sqrt{5}}{2}$$

Lec 4 - Jan 15th 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall: $(a_n) \subseteq \mathbb{R}$ we say (a_n) is **strongly Cauchy** if $\exists (\epsilon_n) \subseteq [0, \infty)$ s.t.

$$\textcircled{1} \sum_{n=1}^{\infty} \epsilon_n < \infty$$

$$\textcircled{2} \forall n, |a_n - a_{n+1}| < \epsilon_n$$

This is on assignment.

Hint: $\sum_{n=1}^{\infty} a_n = L$, $\underbrace{\sum_{k=1}^n a_k}_{S_n} \xrightarrow{n \rightarrow \infty} L$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \Rightarrow \underbrace{|S_n - L|}_{\left| \sum_{k=n+1}^{\infty} a_k \right|} < \epsilon$$

$\rightarrow$ the fact you can make up a tail end, a series of them, and adding up is arbitrarily small

Proof of Banach Contraction Mapping

Let $A \subseteq \mathbb{R}$ be closed and suppose $\exists C \in [0, 1)$ s.t. $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Take a point $x_0 \in A$

and construct $x_1 = f(x_0)$, $x_2 = f(x_1)$, ..., $x_n = f(x_{n-1}) = f^n(x_0)$ } this is the orbit. $\rightarrow$ we aren't including x_0 in the orbit but it doesn't matter

$$\begin{aligned} \text{For } n \in \mathbb{N} \quad |x_{n+1} - x_n| &= \overset{\text{defn of orbit}}{|f(x_n) - f(x_{n-1})|} \leq \overset{\text{contraction assumption}}{C|x_n - x_{n-1}|} = C|f(x_{n-1}) - f(x_{n-2})| \leq C^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\leq C^n |x_1 - x_0| \end{aligned}$$

$C \in [0, 1)$ this is geometric

Since $\sum_{n=1}^{\infty} C^n |x_1 - x_0|$ is a convergent geometric series, we have that (x_n) is strongly Cauchy. Hence $x_n \rightarrow a$ for some $a \in A$. $\hookrightarrow$ This is because domain is closed

Since f is continuous, $\underbrace{f(x_n)}_{x_{n+1}} \rightarrow f(a) \therefore f(a) = a \rightarrow \exists C \quad x_n \rightarrow a, f(x_n) \rightarrow f(a) \Rightarrow f(a) = a$
 $\rightarrow$ What's significant about this?

Suppose $a, b \in A$, s.t. $f(a) = a$ and $f(b) = b$. Then $|f(a) - f(b)| \leq C|a - b| \Rightarrow |a - b| \leq C|a - b|$. Since $C < 1$,

$|a - b| = 0$ and so $a = b$. This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

Chapter 2 Graphical Analysis.

to Visualize the orbit of a under f :

$$\textcircled{1} \text{ Superimpose } y = f(x), y = x$$

② Use a vertical line: $(a, a) \text{ --- } (a, f(a))$

③ Use a horizontal line: $(a, f(a)) \text{ --- } (f(a), f(a))$

④ Vertical line: $(f(a), f(a)) \text{ --- } (f(a), f^2(a))$

⑤ Use a horizontal line: $(f(a), f^2(a)) \text{ --- } (f^2(a), f^2(a))$

Etc...

Ex 1 Using online tool and $f(x) = x^2 - x + 1$, fixed points $x=1$

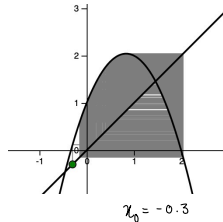
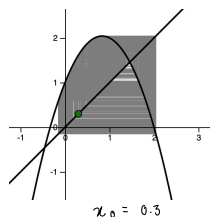
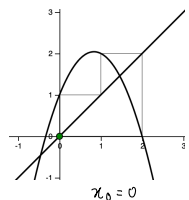
Orbit Analysis:

① $x \in [0, 1]$ we have, $f^n(x) \rightarrow 1$

② otherwise, $x \notin [0, 1]$ $f^n(x) \rightarrow \infty$ In assignment, only expecting this sort of an answer.

Lec 5 - Jan 17th 2024

For $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$ at $x=0$ we have an orbit of 0, 1, 2 which is periodic and graphical is a cycle



points near 0, like 0.06 show chaos. The graphical analysis look does show chaos. Chaos also shows density. We can never hit ever point in the interval b/c an orbit is countable but $[1, 2]$ in $\mathbb{R}$ is uncountable. It is dense [ex. rational numbers is dense in $[0, 1] \in \mathbb{R}$]. We will define dense in a bit.

Chapter 3 - fixed points

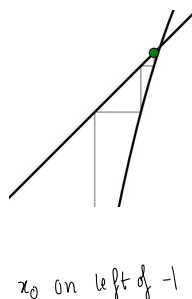
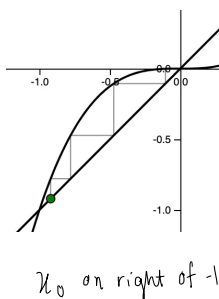
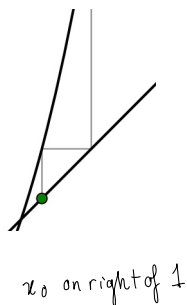
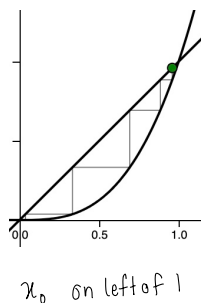
Remark. If $f(x)$ is continuous and $f^n(a) \rightarrow L$, then $f^{n+1}(a) \rightarrow f(L) \rightarrow$ comes from continuity. We also saw this in the B.C.M theorem
 $\therefore f(L) = L$

Motivating Example

$$F(x) = x^3, \text{ fixed points: } 0, \pm 1$$

① for $x \in (-1, 1)$ we see that $f^n(x) \rightarrow 0$. This is an example of an attracting fixed point

② $x \in (1, \infty)$, $f^n(x) \rightarrow \infty$ and $x \in (-\infty, -1)$, $f^n(x) \rightarrow -\infty$. we call ± 1 repelling fixed points



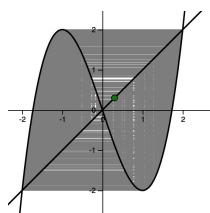
Motivating Example 2

$$f(x) = x^3 - 3x, \text{ fixed points: } 0, \pm 2$$

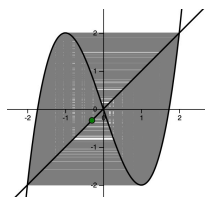
① 0 is repelling (in a different way than last example b/c right & left to 0 is chaotic)

② to the right of 2 the orbits go to infinity, left of -2 orbits go to $-\infty$. so ± 2 is also repelling.

③ at $x_0 = 1$, the orbit is eventually fixed to the fixed point -2; for $x_0 = -1$, the orbit is ev



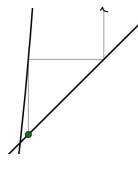
$x_0 = 0.3$



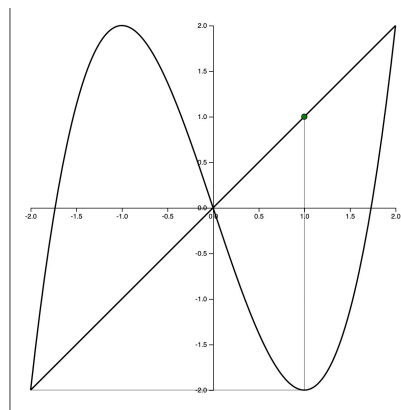
$x_0 = -0.3$



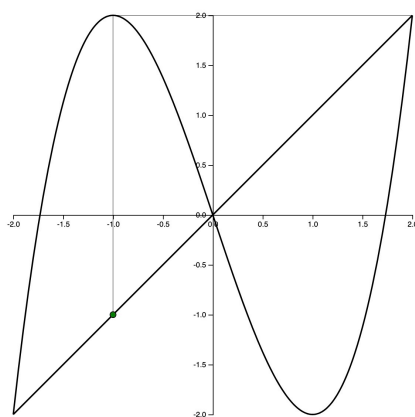
x_0 on left of -2



x_0 on right of 2



orbit of $x_0 = 1$



orbit of $x_0 = -1$

Defn Let $\underline{a}$ be a fixed point of $f(x)$

① if $|f'(\underline{a})| > 1$, we call $\underline{a}$ a repelling fixed point.

② if $|f'(\underline{a})| < 1$, we call $\underline{a}$ an attracting fixed point

③ if $|f'(\underline{a})| = 1$, we call $\underline{a}$ a neutral fixed point $\rightarrow$ this can mean a whole bunch of things

Theorem [Attracting fixed point thm]

Suppose $\underline{a}$ is an attracting fixed of $f(x)$ then $\exists$ an open interval I such that $\underline{a} \in I$ and

① $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I \rightarrow$ slightly unnecessary because $f(x) \in I \Rightarrow f^n(x) \in I$

② $\forall x \in I, f^n(x) \rightarrow \underline{a}$.

Defn: $f: A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$.

① We say $a \in A$ is non-isolated iff $\forall \varepsilon > 0$, $\exists b \in A$ with $b \neq a$ s.t. $b \in (a - \varepsilon, a + \varepsilon)$

② Let $a \in A$ be non-isolated. We say $\lim_{x \rightarrow a} f(x) = L$ iff $\forall \varepsilon > 0 \exists \delta > 0$, $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x - a| < \delta$

$\rightarrow$ if a was an isolated. we could choose a δ where $|x - a| < \delta$ is false. Leading to a false hypothesis which leads to an always true answer and hence anything could be the limit.

Ex $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$ 0 is non isolated

Ex $\{1\} \cup (2, 3)$ 1 is isolated

Proof: [Attracting fixed point thm] $\rightarrow$ to continue in Lec 6

Assume $|f'(a)| < 1$. Then $\exists c \in \mathbb{R}$ s.t. $|f'(a)| < c < 1$

$$\therefore \lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < c$$

Thus $\exists \delta > 0$ s.t. $\frac{|f(x) - f(a)|}{|x - a|} \leq c \quad \forall x \in (a - \delta, a + \delta)$

Hence, for $x \in (a - \delta, a + \delta)$ $|f(x) - f(a)| \leq c|x - a| \rightarrow$ contracting

Lec 6th - Jan 19th 2024

First assignment due 11:59 on Tuesday

Theorem [Attracting fixed point thm]

Suppose a is an attracting fixed point of $f(x)$. There exists an open interval I with $a \in I$ s.t.

① $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I \rightarrow$ saying $f(x) \in I$ is enough but we are choosing to

② $\forall x \in I, f^n(x) \rightarrow a$ write this to give a fuller picture

a is not a boundary pt.
b/c is differentiable due to defn of attracting

b/c this exists, we can choose a maximal δ such that all orbits that go to a exist in it and no other orbits

we have defined the I that was stated in the thm

means f is a contraction

Proof:

Say $|f'(a)| < c < 1$ so that $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < c$. Thus, $\exists \delta > 0$ s.t. $\forall x \in I := (a - \delta, a + \delta), |f(x) - f(a)| \leq c|x - a|$

In particular, for $x \in I$ $|f(x) - f(a)| = |f(x) - a| \leq c|x - a| \leq |x - a| < \delta \Rightarrow f(x) \in I$

Continuing, for $x \in I$ $|f^n(x) - a| \leq c^n|x - a| \leq |x - a| < \delta$ so that $f^n(x) \in I$. Finally, for $x \in I$ $0 \leq |f^n(x) - a| \leq c^n|x - a| \rightarrow 0$

Note: If the question asks to use a ϵ - δ proof then do so.

Can show using ϵ - δ proof if needed

Theorem [repelling fixed point theorem]

Suppose a is a repelling fixed point for $f(x)$. There exist an open interval $a \in I$, s.t. $\forall x \in I, x \neq a, \exists n \in \mathbb{N}$ s.t. $f^n(x) \notin I$
a repelling point need not shoot off to infinite, it may be chaotic where for some $n, f^n(x) \notin I$

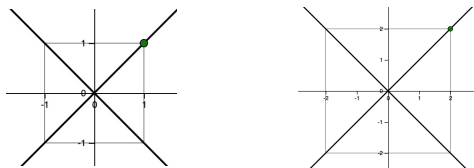
Proof: Say $|f'(a)| > c > 1$. then $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} > c$ and so $\exists \delta > 0$ s.t. for all $x \in I := (a - \delta, a + \delta)$ $|f(x) - f(a)| \geq c|x - a|$

Since a is a fixed point, $|f(x) - f(a)| = |f(x) - a|$. Suppose $\forall n, f^n(x) \in I$. As before $|f^n(x) - a| \geq c^n|x - a| \rightarrow \infty$ This infinity goes well beyond delta!

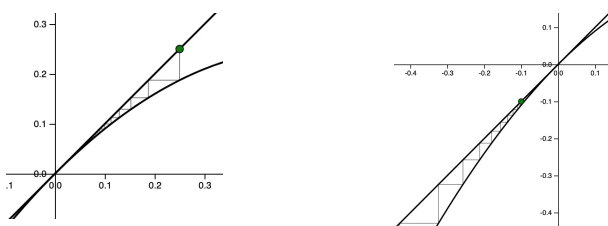
What is a neutral fixed point? A lot.

Investigation: Neutral fixed points

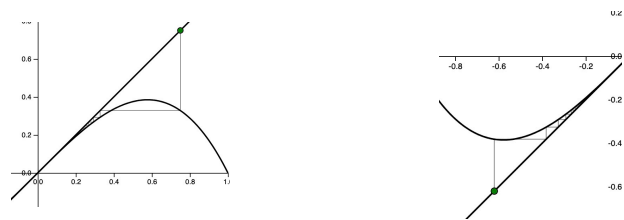
Ex $f(x) = -x$, 0 is a fixed point and $|f'(0)| = 1$. bounces around



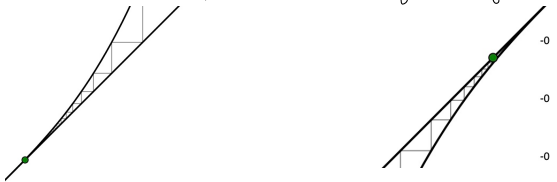
Ex $f(x) = x - x^2$, $|f'(0)| = 1$ attracting to the right and repelling to the left.



Ex $f(x) = x - x^3$ $|f'(0)| = 1$, this is weakly attracting, attracting but too slowly, happens in asymptote situations

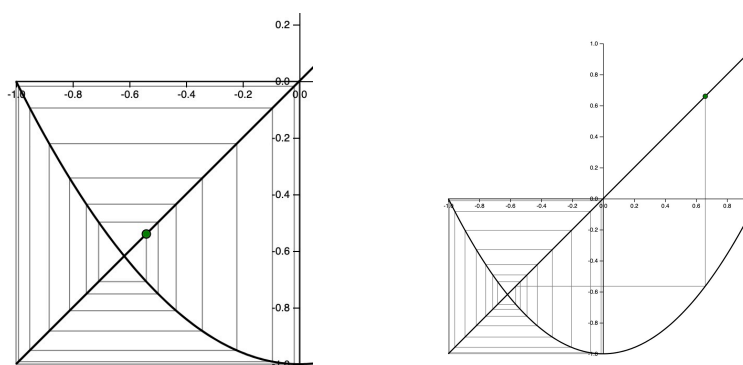


Ex $f(x) = x + x^3$, $|f'(0)| = 1$, weakly repelling, repelling too slow



Motivating Example: $f(x) = x^2 - 1$, $a = 0$.

Orbit, $(0, -1, 0, -1, 0, -1)$; 0 is a periodic point of period 2. x_0 near 0 shows orbits getting closer to $(0, 1)$ -cycle. we will call 0 an attracting periodic point because. 0 is an attracting fixed point of $f^2(x)$.



Lec 7 - Jan 22nd 2024

Defn: Let a be a **periodic point** for $f(x)$ with period n . We say a is **an attracting / repelling / neutral** per point for $f(x)$ iff a is an attracting / repelling / neutral fixed point of $f^n(x)$.

for strictly periodic orbits. [no eventually periodic] the points in the periodic orbit are fixed points of $f^n(x)$. Also, every point in the orbit will have the same classification as the point tested. $\rightarrow$ by proposition value of derivative must be the same for any point in the cycle.

Prop: Let $f(x)$ be a function. Then
 $\hookrightarrow f(x)$ need to be diffable

$$(f^n)'(a) = f'(a) f'(f(a)) f'(f^2(a)) \dots f'(f^{n-1}(a))$$

Proof: [Induction]

$$\text{if } n=1, f'(a) = f'(a)$$

Assume the result holds for some $n, n \geq 1$.

$$\text{then } \frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

$$\text{Then } (f^{n+1})'(a) = \underbrace{f'(f^n(a)) \cdot f'(a) \cdot f'(f(a)) \dots f'(f^{n-1}(a))}_{\text{inductive hypothesis}} \quad \blacksquare$$

$$\text{Ex) } f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1, a=0, \text{ orbit } (0, 1, 2, 0, \dots)$$

$$f'(x) = -3x + \frac{5}{2}$$

$$\begin{aligned} (f^3)'(0) &= f'(0) f'(1) f'(2) \\ &= \left(-\frac{7}{2}\right) \left(-\frac{1}{2}\right) \left(\frac{5}{2}\right) \\ &= \frac{35}{8} > 1 \quad \therefore \text{repelling} \end{aligned}$$

Chapter 4. Bifurcations

In general Bifurcation theory is the study of how a family of curves can change when a defining parameter is change

We will consider the quadratic family: $Q_c(x) = x^2 + c$ $\hookrightarrow$ very important family
Idea: how does c affect the dynamics of $Q_c(x)$?

to find the fixed points of $Q_c(x)$:

$$x^2 + c = x \iff x^2 - x + c = 0$$

$$\iff x = \frac{1 \pm \sqrt{1 - 4c}}{2}$$

Thus:

① $Q_c(x)$ has 2 fixed points when $c < \frac{1}{4}$

② $Q_c(x)$ has 1 fixed point when $c = \frac{1}{4}$.

③ $Q_c(x)$ has no fixed points [real] when $c > \frac{1}{4}$.

If $c > \frac{1}{4}$, $Q_c^n(x)$ must diverge to ∞ .

If $c = \frac{1}{4}$, $Q_c(x)$ has the unique fixed point of $p = \frac{1}{2}$. Since $Q'_c(x) = 2x$, $Q'_c(\frac{1}{2}) = 1$ so this fixed point is neutral → attracting to left and repelling to right

If $c < \frac{1}{4}$, $Q_c(x)$ has 2 fixed points:

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad p_- = \frac{1 - \sqrt{1-4c}}{2}$$

First,

$$Q'_c(p_+) = 1 + \sqrt{1-4c} > 1 \Rightarrow p_+ \text{ is repelling}$$

Next,

$$-1 < Q'_c(p_-) < 1$$

$$\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1$$

$$\Leftrightarrow -2 < -\sqrt{1-4c} < 0$$

$$\Leftrightarrow 2 > \sqrt{1-4c} > 0$$

$$\Leftrightarrow -\frac{3}{4} < c < \frac{1}{4} \rightarrow \text{this is the range for attracting fixed point}$$

$$\text{if } c < -\frac{3}{4}, \quad Q'_c(p_-) < -1$$

$$\text{if } c = -\frac{3}{4}, \quad Q'_c(p_-) = -1$$

Theorem: For the family $Q_c(x) = x^2 + c$:

① All orbits tend to ∞ if $c > \frac{1}{4}$

② when $c = \frac{1}{4}$, $Q_c(x)$ has a unique fixed point, $\frac{1}{2}$, and it is neutral

③ if $c < \frac{1}{4}$, $Q_c(x)$ has 2 fixed points p_+ and p_- . The point p_+ is repelling

Moreover

a) If $-\frac{3}{4} < c < \frac{1}{4}$, p_- is attracting

b) if $c = -\frac{3}{4}$, p_- is neutral

c) if $c < -\frac{3}{4}$, p_- is repelling

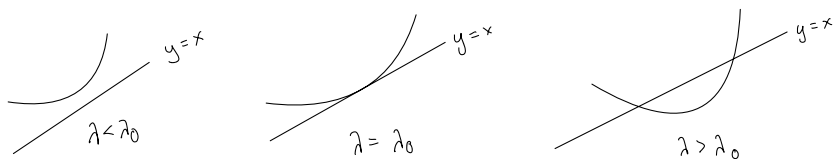
Lec 8 - Jan 24 2024

Def'n: We say a family of functions $F_\lambda(x)$ undergoes a **bifurcation** at λ_0 if there is a change in fixed point structure at λ_0 .

Ex) $Q_c(x) = x^2 + c$, $\lambda_0 = \frac{1}{4}$

Def'n: A family $F_\lambda(x)$ undergoes a **tangent bifurcation** at λ_0 if there is an open interval I and $\epsilon > 0$ s.t.

- ① For $\lambda_0 - \epsilon < \lambda < \lambda_0$, $F_\lambda(x)$ has no fixed points on I
 - ② For $\lambda = \lambda_0$, $F_\lambda(x)$ has one fixed point and it is neutral
 - ③ For $\lambda_0 < \lambda < \lambda_0 + \epsilon$, $F_\lambda(x)$ has 2 fixed points in I , one attracting and the other repelling
- or ①, ②, ③ with $>$.



Ex) $E_\lambda(x) = e^x + \lambda$, $\lambda_0 = -1$. This is an example of a tangent bifurcation.

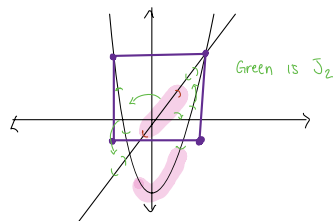
Ex) $F_\lambda(x) = \lambda x(1-x)$, $\lambda_0 = 1$. Not tangent bifurcation.

Chapter 5 Cantor Set

Investigation: $Q_c(x) = x^2 + c$, $c < -2$.

$$p_+ = \frac{1 + \sqrt{1-4c}}{2} > 2, \quad -p_+ < -2$$

Consider $I = [-p_+, p_+]$ and $I \times I$



J_2 is asking which x has y values in J_1 , which is how these intervals are

, picture not to scale

Let $J_i \subseteq I$ be the interval s.t. $Q_c(x) \notin I$ for all $x \in J_i$,

For $x \in J_1$, $Q_c^n(x) \rightarrow \infty$. Moreover, If $\exists n$ s.t. $Q_c^n(x) \in J_1$ then $Q_c^n(x) \rightarrow \infty$

Consider $\Lambda = \{x \in I : Q_c^n(x) \in I \forall n\}$

Big Idea: Λ contains all the points with interesting orbits.

$$J_1 = \{x \in I : Q_c(x) \notin I\}$$

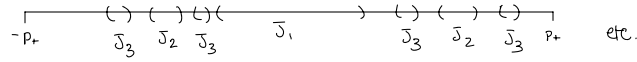
$$J_2 = \{x \in I : Q_c(x) \in J_2\} = \{x \in I : Q_c^2(x) \notin I\}$$

$$J_3, J_4, J_5 \dots$$

what is a cantor set?

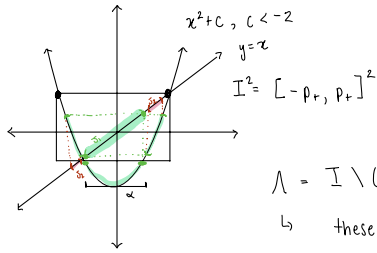
Then, $\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$ is a cantor set. Drawing Λ on the x -axis:

Symmetry comes from the curve (parabola)



It is a fractal
b/c of its self repeating nature!

Lec 9 - Jan 26th 2024



One quarter mark for the course.

$$\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$$

$$A = \{x \in I : \forall n, Q_n^+(x) \in I\}$$

↳ these are all the interesting orbits

↳ Cantor set

Construction: Cantor Middle thirds set

$C_0 = [0, 1]$ → remove open middle third interval

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

⋮

$K = \bigcap_{n=1}^{\infty} C_n$ → Cantor [middle-third] set. ↑ these are nested A cantor set is taking out a chunk of the interval, infinitely. We can see that.

the end points will always survive.

Proposition: $A_n \subseteq \mathbb{R}$ closed. Then $\bigcap_{n=1}^{\infty} A_n$ is closed.

Why? $(a_k) \in \bigcap A_n$, $a_k \rightarrow a$. $\forall n$, $(a_k) \subseteq A_n \Rightarrow a \in A_n \Rightarrow a \in \bigcap A_n$

Proposition: $A, B \subseteq \mathbb{R}$ closed. Then $A \cup B$ is closed.

Why? $(a_n) \in A \cup B$, $a_n \rightarrow a$. WLOG. $\{n : a_n \in A\}$ is infinite. This allows us to construct $(b_n) \in A$ s.t. $b_n \rightarrow a$. Since A is closed, $a \in A \subseteq A \cup B$.

Theorem K is closed [in fact all Cantor sets are] → combine the 2 propositions above to get this

Theorem: K contains no non-empty open intervals → Hausdorff property of $\mathbb{R}$?

Why? $I \subseteq K \Rightarrow \forall n, I \subseteq C_n \Rightarrow \forall n, \ell(I) \leq \frac{1}{3^n} \Rightarrow \ell(I) = 0 \Rightarrow I = \emptyset$ which is a contradiction.
↳ length of interval.

no isolated points
interior points?

Lets consider the base -3 expansion of $x \in [0, 1]$. $x = 0.s_1s_2s_3\dots$, $s_i \in [0, 1, 2]$

$$\begin{array}{l} s_1 = 0 \\ [0, \frac{1}{3}] \\ s_1 = 0, s_2 = 0, s_3 = 0 \text{ etc.} \\ [0, \frac{1}{9}] \end{array} \quad \begin{array}{l} s_1 = 1 \\ [\frac{2}{9}, \frac{1}{3}] \\ s_1 = 1, s_2 = 1, s_3 = 1 \text{ etc.} \\ [\frac{2}{9}, \frac{7}{9}] \end{array} \quad \begin{array}{l} s_1 = 2 \\ [\frac{8}{9}, 1] \\ s_1 = 2, s_2 = 2, s_3 = 2 \text{ etc.} \\ [\frac{8}{9}, 1] \end{array}$$

Fact:

$x \in K$ iff x can be written in base-3 using only 0's and 2's

Ex $\frac{1}{3} \in K$. $\frac{1}{3} = [0.1]_3 = [0.02222\ldots]_3 \rightarrow \# \text{ theory? base } \alpha?$

Theorem: K is uncountable $|K| = |\mathbb{R}| \rightarrow \text{cardinality} \rightarrow \text{not important to this course, just for interest.}$

How to remember? 0 means left, 2 means right.

Start Ch 6 on Monday.

Lec 10 - Jan 29th 2024

Chapter 6 - Symbolic Dynamics

Recall: $Q_c(x) = x^2 + c, \quad c < -2$

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad I = [-p_+, p_+]$$

$$J_1 = \{x \in I : Q_c(x) \in I\}$$

$$J_2 = \{x \in I : Q_c(x) \in J_1\}$$

$$J_3 = \{x \in I : Q_c(x) \in J_2\}$$

$\vdots$

$\Lambda = I \setminus (\cup J_i) = \{x \in I : \forall n, Q_c^n(x) \in I\} \rightarrow$ This is a Cantor set. All Cantor sets are isomorphic to each other ↗ verify iso vs. homeo

Notation:

$$I \setminus J_1 = I_0 \cup I_1$$

$I = [I_0 \text{ removed}] [I_1]$ → Idea we have a left part and a right part.

Defn: For $x \in \Lambda$, we define the itinerary of x to be the sequence

→ kind telling you direction, dealing with orbit indirectly

$$S(x) = (x_0 x_1 x_2 x_3 \dots), \quad x_i \in \{0, 1\}$$

where $x_i = 0 \iff Q_c^i(x) \in I_0$

$$x_i = 1 \iff Q_c^i(x) \in I_1$$

Goal: Understand $S(x)$ better

Notation: $\Sigma^1 = \{(x_0 x_1 x_2 \dots) : x_i \in \{0, 1\}\} \rightarrow$ we drop the commas?

→ function, specifically a transform. Claims it's easier to work with.

$$S: \Lambda \rightarrow \Sigma^1$$

↪ S is a homeomorphism $\Rightarrow \Lambda$ is same as Σ^1 , what we are doing this week.

Defn: Let X be a set. We say $d: X \times X \rightarrow [0, \infty)$ is a Metric iff

① $d(x, y) = 0$ iff $x = y$

② $\forall x, y \in X, \quad d(x, y) = d(y, x)$

③ $\forall x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(z, y) \rightarrow \text{triangle inequality}$$

We call (X, d) a metric space

Idea $d(x, y) =$ distance between x and y .

Ex) $X = \mathbb{R}$, $d(x, y) = |x - y|$

Ex) $X = \mathbb{R}^n$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

Ex) X any set, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ discrete metric

These are all Metric space

Ex) $A \subseteq \mathbb{R}$, For $x, y \in A$, $d(x, y) = |x - y|$ is a metric (A, d)

Ex) [Cantor Space]

$X = \Sigma$, $d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$
 $\downarrow \quad \downarrow$
 $x_0, x_1, \dots \quad y_0, y_1, \dots$
ensuring we get a number b/c it gives us a convergent series

Ex) $x = (1 \ 1 \ 1 \dots)$
 $y = (1 \ 0 \ 1 \ 0 \dots)$

$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \rightarrow 1^s$ cancel in every even step

How $\rightarrow$
 $= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} = \sum_{i=0}^{\infty} \left(\frac{1}{2^{2i}}\right) \frac{1}{2}$
 $= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{1}{2} \left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3}$

Proposition: $x, y \in \Sigma$

① if $x_i = y_i$ for $i \leq n$ then $d(x, y) \leq \frac{1}{2^n}$

First few terms carry a lot of weight

② if $d(x, y) < \frac{1}{2^n}$ then $x_i = y_i$ for $i \leq n$

Why?

① $d(x, y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1}{2^n} \rightarrow$ convergence of geometric series formula?

② $d(x, y) < \frac{1}{2^n}$

$\exists k \leq n$, $x_k \neq y_k$. $\Rightarrow |x_k - y_k| = 1$
 $d(x, y) \geq \frac{1}{2^k} \geq \frac{1}{2^n}$ hence contradiction

Ex) $x = (000\dots)$

$y = (1000\dots)$

$d(x, y) = \frac{1}{2^0} = 1$ BUT $x_0 \neq y_0$ This shows that Proposition cannot be iff.

Defn: Shift map.

$\sigma: \Sigma \rightarrow \Sigma$

$\sigma: (x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$

Remark $\sigma^k(x_0, x_1, x_2, \dots) = (x_k, x_{k+1}, x_{k+2}, \dots)$

Connections

- dynamical systems on manifolds
- p-adic numbers? Veritasium video

Whats next?

Show ① $\sigma: \Sigma \rightarrow \Sigma$ is continuous ↪ gives us

② $\sigma: \Sigma \rightarrow \Sigma$ dynamical System

③ $S: \Lambda \rightarrow \Sigma$ homeomorphism $\rightarrow$ means Λ and Σ are "same space"

④ $\sigma: \Sigma \rightarrow \Sigma$ Chaotic $\Rightarrow Q_c: \Lambda \rightarrow \Lambda$

Lec 11 - Jan 31 2024

Recall: $Q_c(x) = x^2 + c$, $c < -2$

$$\Lambda = \{x \in I : \forall n, Q_c^n(x) \in I\}$$

$S: \Lambda \rightarrow \Sigma$, $S(x) = (x_0, x_1, x_2, \dots)$ iff $Q_c^i(x) \in I_{x_i}$

either x_1 or x_0

is this extendable to other iterations in the Cantor Set.

if d' and d where abs then we get $\uparrow$ continuous defn for $f: \mathbb{R} \rightarrow \mathbb{R}$

Def'n: (X, d) , (Y, d')

We say $f: X \rightarrow Y$ is continuous at $y \in X$ iff $\forall \epsilon > 0, \exists \delta > 0, x \in X, d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$

we say $f: X \rightarrow Y$ is continuous iff it is continuous at every $y \in X$

Proposition: $\sigma: \Sigma \rightarrow \Sigma$ is continuous

Proof: Fix $y \in \Sigma$ and let $\epsilon > 0$ be given. Say $y = (y_0, y_1, y_2, \dots)$. Take $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \epsilon$.

Consider $\delta = \frac{1}{2^{n+1}}$. Let $x = (x_0, x_1, x_2, \dots) \in \Sigma$ s.t. $d(x, y) < \delta$. $\therefore x_i = y_i$ for $i = 0, 1, \dots, n+1$

Then, $\sigma(x) = x_1, x_2, x_3, \dots$, $\sigma(y) = y_1, y_2, y_3, \dots$ and $x_i = y_i$ for $i = 1, 2, \dots, n+1$.

By Previous Prop $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \epsilon$

Def'n: (X, d) , $(x_n) \subseteq X$, $x \in X$. we say (x_n) converges to x , $x_n \rightarrow x$, iff $\forall \epsilon > 0, \exists N \in \mathbb{N}, n > N \Rightarrow d(x_n, x) < \epsilon$

Proposition (X, d) , (Y, d') , $f: X \rightarrow Y$. Then, f is continuous iff $f(x_n) \rightarrow f(x)$ whenever $x_n \rightarrow x$

Def'n: (X, d) , (Y, d') , $f: X \rightarrow Y$, we call f a homeomorphism iff

- ① f is injective (one to one)
- ② f is surjective
- ③ f is continuous
- ④ f^{-1} is continuous

Investigation: $f: X \rightarrow Y$ is a homeomorphism

① $(x_n) \subseteq X$, $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$

by continuity of f^{-1}

② $(y_n) \subseteq Y$, $y_n \rightarrow y$. Say $y_n = f(x_n)$, $y = f(x) \Rightarrow f(x_n) \rightarrow f(x) \Rightarrow f^{-1}(f(x_n)) \rightarrow f^{-1}(f(x)) \Rightarrow x_n \rightarrow x$

Big Idea: f is a relabelling of X to Y . we think of X and Y as the "same" metric space

Homeomorphic spaces are basically equal.

Remark

① $x \in \Lambda$, $S(x) = (x_0, x_1, \dots)$

We know that $x_0 \in I_{x_0}$, $x_1 \in I_{x_1}$, $Q_c^2(x) \in I_{x_2}, \dots$

$\therefore S(Q_c(x)) = (x, x_2, \dots) = \sigma(S(x))$

② $S(Q_c^n(x)) = \sigma^n(S(x)) \rightarrow$ These 2 are friends and work together

Theorem: $S: \Lambda \rightarrow \Sigma^+$ is a homeomorphism.

Aim is to define
and prove chaos

Theorem: Monotone Convergence theorem

If $(a_n) \subseteq \mathbb{R}$ is increasing / decreasing and bounded. then (a_n) converges.

Theorem: [Nested Intervals Lemma]

If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are closed intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ $\rightarrow$ open intervals could.

Idea: $I_k = [a_k, b_k]$

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \dots$

(a_n) increasing, $(a_n) \subseteq [a_1, b_1]$

(b_n) decreasing, $(b_n) \subseteq [a_1, b_1]$

$\rightarrow$ they converge

By MCT, $a_n \rightarrow a$ and $b_n \rightarrow b$ $\therefore [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n \rightarrow$ worst case scenario $a = b$

Lec 12th 2024 - Feb 2nd 2024

Thm: $S: \Lambda \rightarrow \Sigma$ is a homeomorphism. → what's the difference between transforms and homeomorphism

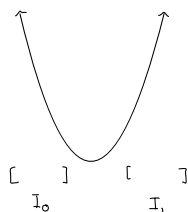
Remark: This is true for $c < -2$. We will prove the result for $c < \frac{-(5+2\sqrt{5})}{4}$.

A2: $\forall x \in I \setminus J_i = I_0 \cup I_1, |Q'_c(x)| \geq \mu > 1$. μ exists by our A2 Q for the above c values
↓
First step in Cantor process / itinerary

Proof:

Lets first show it is injective

Suppose $x, y \in \Lambda$ with $S(x) = S(y)$ but $x \neq y$. Therefore $\forall n, Q_c^n(x)$ and $Q_c^n(y)$ live in the same I_0 and I_1 .



By the Mean Value Theorem

$$|Q_c(x) - Q_c(y)| \geq \mu |x - y|$$

parabola is injective on small intervals

Since Q_c is injective on I_0 and I_1 , we have that $Q_c(x) \neq Q_c(y)$. Thus,

$$\begin{aligned} |Q_c^2(x) - Q_c^2(y)| &\geq \mu^2 |x - y| \\ &\vdots \\ |Q_c^n(x) - Q_c^n(y)| &\geq \mu^n |x - y| \end{aligned}$$

Since $\mu > 1$, $\mu^n |x - y| \rightarrow \infty$. However,

$$|Q_c^n(x) - Q_c^n(y)| \leq \max \{ \overset{\text{length}}{\ell(I_0)}, \ell(I_1) \} \rightarrow \text{length will be the same.}$$

contradiction!

Surjective: Let $y = (y_0, y_1, y_2, \dots) \in \Sigma$. For $n \in \mathbb{N}$, Let $I_{y_0, y_1, \dots, y_n} = \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \dots, Q_c^n(x) \in I_{y_n}\}$.

It is enough to show that there exist $x \in \bigcap_{n=1}^{\infty} I_{y_0, y_1, \dots, y_n}$ ($\Rightarrow S(x) = y$)

Nested Interval Lemma
Can apply if this is an interval

Clearly, $I_{y_0} \supseteq I_{y_0, y_1} \supseteq I_{y_0, y_1, y_2} \supseteq \dots$

Claim: For all $n \geq 0$ $I_{y_0, y_1, \dots, y_n}$ is a closed interval. First, $I_{y_0} \in \{I_0, I_1\}$ is a closed interval. Assume $I_{y_0, \dots, y_n}$ is a closed interval for some $n \geq 0$.

Note: $x \in I_{y_0 y_1 \dots y_n y_{n+1}} \iff x \in I_{y_0}$

$$f: X \rightarrow Y, \quad B \subseteq Y$$

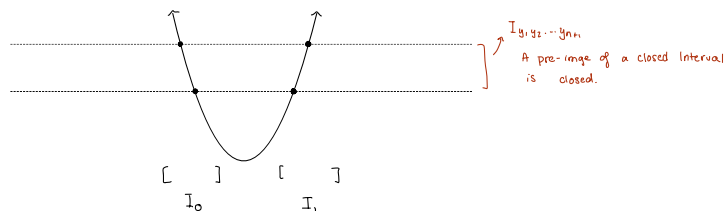
$$f^{-1}(B) = \{x \in X: f(x) \in B\}$$

$$Q_c(z) \in I_{y_1}, \quad Q_c(Q_c(z)) \in I_{y_2}, \quad Q_c(Q_c^2(z)) \in I_{y_3}, \dots$$

$$(*) \iff I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \dots y_n}) \rightarrow \text{similar to how shift map relates}$$

↗ means pre-image

By induction, $I_{y_1 y_2 \dots y_{n+1}}$ is closed interval. We have $I_{y_1 y_2 \dots y_n} \subseteq I_{y_1} \in \{I_0, I_1\}$



$Q_c^{-1}(I_{y_1 \dots y_{n+1}})$ is a union of two disjoint closed intervals, one in I_0 , and one in I_1 . Returning to $(*)$

$$I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \dots y_{n+1}})$$

is one of these closed intervals. This proves the claim. By the Nested Intervals Lemma, $\exists x \in \bigcap_{n=1}^{\infty} I_{y_0 \dots y_n}$. Hence $S(x) = y$ and so S is surjective

Continuity

Fix $y \in \Lambda$ and say $S(y) = (y_0 y_1 y_2 \dots)$. Let $\epsilon > 0$ be given and choose n such that $\frac{1}{2^n} < \epsilon$. Consider the 2^{n+1}

disjoint closed intervals $I_{t_0 t_1 \dots t_n}$. Pick $\delta > 0$ such that $(y - \delta, y + \delta)$ only overlaps with $I_{y_0 y_1 \dots y_n}$. Note: $y \in I_{y_0 y_1 \dots y_n}$

For $x \in \Lambda$ with $|x - y| < \delta$, $x \in I_{y_0 y_1 \dots y_n}$ and so ↗ means first n terms agree

$$d(S(x), S(y)) \leq \frac{1}{2^n} < \epsilon$$

Continuity of S^{-1}

Similar



Lec 13 - Feb 5th 2024

Week 5. 6 weeks before Reading week. 6 weeks after Reading week. Nothing Due this week. Midterm March 1. Midterm will be in a different room no questions after class this week.

Chapter 7 - Chaos

Def'n: (X, d) , we say $A \subseteq X$ is **dense** iff $\forall x \in X, \forall \epsilon > 0, \exists a \in A$ s.t. $d(a, x) < \epsilon$

Ex. $\mathbb{Q}$ is dense in $\mathbb{R}$. $\rightarrow$ easiest way to do so is by Decimal expansion and Cut-off at a certain # of decimal points

Ex. $\mathbb{Z}$ is not dense in $\mathbb{R}$.

Ex. $A = \{x \in \Sigma^1 : \exists N, \forall i > N, x_i = 0\} \rightarrow$ sequence of eventually all zeros This is Dense in Σ^1

Proof: Let $x \in \Sigma^1$ and let $\epsilon > 0$ be given. Take $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \epsilon$. Consider $y = (x_0 x_1 \dots x_n 0 0 \dots) \in A$. Then $d(x, y) < \frac{1}{2^n} < \epsilon$ ■

$\rightarrow$ SUPER IMPORTANT TO DO
Ex. $A = \{x \in \Sigma^1 : x \text{ periodic}\}$ is dense in Σ^1 hint $y = (x_0 x_1 \dots x_n x_0 x_1 \dots x_n \dots)$ where $x \in \Sigma^1$ and $x \geq y$ agree up to the first n -terms

Remark $A = \{x \in \Sigma^1 : x \text{ periodic}\}$ is exactly the periodic points for $\sigma: \Sigma^1 \rightarrow \Sigma^1$ $\rightarrow$ Shift map.

Prop: There exists $z \in \Sigma^1$ s.t. $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Σ^1 . $\rightarrow$ There could be multiple z . but we only need 1.

Proof: Take $z = (01000100110000010010011\dots)$ $\rightarrow$ writing out all combinations of 01 blocks for all possible lengths. This ensures any $x \in \Sigma^1$ will be close

Take $x \in \Sigma^1$ and $\epsilon > 0$. Let $\frac{1}{2^n} < \epsilon$. $\rightarrow$ k allows you to shift to the right block. The bigger the n , the more chopping. For some $k, \sigma^k(z)$ and x agree on the first n terms. Thus $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \epsilon$ ■

Def'n: A **Dynamical System** is a metric space (X, d) together with a continuous function $f: X \rightarrow X$. This allows you to do what we have done on any metric space

Notation: $f: X \rightarrow X$ dynamical System.

Ex) $\sigma: \Sigma^1 \rightarrow \Sigma^1$ is a dynamical System.

$\rightarrow$ gross?
Def'n: Let $f: X \rightarrow X$ be a dynamical System. We say $f: X \rightarrow X$ is **transitive** iff $\forall x, y \in X, \forall \epsilon > 0, \exists z \in X, n, m \in \mathbb{N} \cup \{0\}$ s.t. $d(x, f^n(z)) < \epsilon$ and $d(y, f^m(z)) < \epsilon$ No matter which 2 points you start with, There is an orbit of a point that is close to the original 2 points.

Proposition: $\sigma: \Sigma^1 \rightarrow \Sigma^1$ is transitive

Proof: take $z \in \Sigma^1$ s.t. $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Σ^1 . For all $\epsilon > 0, x, y \in \Sigma^1, \exists n, m \in \mathbb{N} \cup \{0\}$ s.t. $d(x, \sigma^n(z)) < \epsilon$ and $d(y, \sigma^m(z)) < \epsilon$ ■
 $\hookrightarrow$ having a dense set makes transitivity easy.

It doesn't need to be dense in order for the function to be transitive.

$\rightarrow$ worst one today
Def'n: Let $f: X \rightarrow X$ be a Dynamical System. we say $f: X \rightarrow X$ depends sensitively on initial conditions (**is sensitive**) iff $\exists \beta > 0, \forall \epsilon > 0, \forall x \in X, \exists y \in X, \exists k \in \mathbb{N}$ such that $d(x, y) < \beta$ and $d(f^k(x), f^k(y)) \geq \epsilon$ $\rightarrow$ β doesn't depend on $x \geq y$

$$① \quad d(x, y) < \epsilon$$

$$② \quad d(f^k(x), f^k(y)) \geq \beta \quad \rightarrow \text{this is similar to stable/unstable}$$

Prop: $\sigma: \Sigma^1 \rightarrow \Sigma^1$ is sensitive.

Proof: Take $\beta = 1$. Let $\epsilon > 0$ be given and let $x \in \Sigma$. Say $\frac{1}{2^n} < \epsilon$ and suppose $y \in \Sigma$ is st. $0 < d(x, y) < \frac{1}{2^n}$. Thus, $\exists k \geq n$, st $x_k \neq y_k$. $\therefore d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^0} = 1 = \beta$.
 $\rightarrow$ either 0 or 1 so $|x_k - y_k| = 1$.

Lec 14 - Feb 7th 2024

Def'n: A dynamical system $f: X \rightarrow X$ is **chaotic** iff

- ① The periodic points for f are dense in X .
- ② transitive
- ③ Sensitive

Theorem: $\sigma: \Sigma \rightarrow \Sigma$ is chaotic $\rightarrow$ we have already proved this above

$\rightarrow$ Does chaos imply repelling? Prof must think about it.

Proposition: (X, d) , (Y, d') and $f: X \rightarrow Y$ continuous and surjective. $\rightarrow$ allows us to not "kill" A when f is applied. If $A \subseteq X$ is dense in X then $f(A)$ is dense in Y .

Proof: Let $y \in Y$ and say $y = f(x)$. $\rightarrow$ can do b/c of surjective. Let $\epsilon > 0$ be given. Since f is continuous at x , $\rightarrow$ continuous defn. $\exists \delta > 0$, $d(z, x) < \delta \Rightarrow d'(f(z), f(x)) < \epsilon$

Since A is dense in X , we may find $a \in A$ such that $d(a, x) < \delta \Rightarrow$ $\rightarrow$ density defn. $d'(f(a), f(x)) < \epsilon$
 $\Rightarrow d'(f(a), y) < \epsilon$ ■

Theorem: $c < \frac{-(5 + 2\sqrt{5})}{4}$ then $Q_c: \Lambda \rightarrow \Lambda$ is chaotic.

Proof: IDEAS: Relate this theorem to the above thm that $\sigma: \Sigma \rightarrow \Sigma$ is chaotic

① Periodic Points

$\rightarrow$ b/c S is one-to-one.

Note that $Q_c^n(x) = x$ for x that is periodic $\Leftrightarrow S(Q_c^n(x)) = S(x)$
 $\Leftrightarrow \sigma^n(S(x)) = S(x)$

By using the previous proposition with $S^{-1}: \Sigma \rightarrow \Lambda$, the periodic point for Q_c are dense in Λ

② Transitive

Take $z \in \Sigma$ such that $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense. Again, $\{S^{-1}(\sigma^k(z)) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Λ

Note: Say $S(x) = z$. $S(Q_c^k(x)) = \sigma^k(S(x)) \Leftrightarrow Q_c^k(x) = S^{-1}(\sigma^k(S(x)))$. Thus $\{Q_c^k(x) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Λ .

as before, $Q_c: \Lambda \rightarrow \Lambda$ is transitive.

③ Sensitivity

Recall that $\Lambda \subseteq I \setminus J_1 = I_0 \cup I_1$. Let $\beta > 0$ be less than the distance between I_0 and I_1 . For $x, y \in \Lambda$ with $x \neq y$, $S(x) \neq S(y)$

$\Rightarrow \exists k$ such that k^{th} term of $S(x) \neq k^{\text{th}}$ term of $S(y)$. Hence $|Q_c^k(x) - Q_c^k(y)| > \beta$ ■

This is slightly stronger than our defn of Sensitivity

End of Midterm
Material

Lec 15- Feb 9th 2024

Chapter 8 Sarkovskii's Theorem

Theorem: Period Three

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. If f has a point with period 3, then f has point with n for all $n \in \mathbb{N}$.

Proposition 1: $I \subseteq J$ are closed intervals. $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous. If $f(I) \supseteq J$, then $f(x)$ has a fixed point in I .

Why? $[a, b] = I \subseteq J = [c, d]$

$$c = f(p), d = f(q); p, q \in I$$

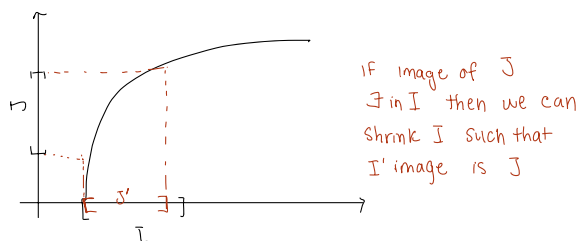
$g(x) = f(x) - x$ is also continuous.

$$g(p) = f(p) - p = c - p \leq 0 \quad \text{left end pt of big interval}$$

$$g(q) = f(q) - q = d - q \geq 0 \quad \text{right end pt of big interval}$$

By IVT, $\Rightarrow \exists z, g(z) = f(z) - z = 0 \Rightarrow f(z) = z \therefore \exists$ a fixed point. ■

Proposition 2: I, J be closed intervals. $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. $f(I) \supseteq J$ then, $\exists$ closed interval $I' \subseteq I$ such that $f(I') = J$



Proof: [Period 3]

Let $a \in \mathbb{R}$ be a period 3 point for $f(x)$. Say $f(a) = b, f(b) = c, f(c) = a$. WLOG, suppose $a < b, c$.

Case 1 $a < b < c$. $\rightarrow$ Case $a < c < b$ similar

Let $I = [a, b]$ and $J = [b, c]$. Observe $f(a) = b$ and $f(b) = c \Rightarrow J \subseteq f(I)$ and $f(b) = c, f(c) = a \xrightarrow{\text{IVT}} I \cup J \subseteq f(J)$

Since $J \subseteq f(J)$ $\exists$ closed interval $A_1 \subseteq J$ such that $f(A_1) = J$. Again, $A_1 \subseteq J = f(A_1)$, and so there exists a closed interval $A_2 \subseteq A_1$

such that $f(A_2) = A_1$.

For $n > 3$. Repeating the above process we can find $A_{n-2} \subseteq A_{n-3} \subseteq \dots \subseteq A_2 \subseteq A_1 \subseteq J$ such that $f(A_i) = A_{i-1}$.

$\downarrow$
 $\text{in } J$

in I
↑

Now $f(I) \supseteq J \supseteq A_{n-2} \Rightarrow \exists A_{n-1} \subseteq I$ closed interval such that $f(A_{n-1}) = A_{n-2}$

Moreover, $f(J) \supseteq I \supseteq A_{n-1} \Rightarrow \exists A_n \subseteq J$ closed interval such that $f(A_n) = A_{n-1}$. We have $f^n(A_n) = J$ and $A_n \subseteq J$. By prop 1,

$\exists x_0 \in A_n$ such that $f^n(x_0) = x_0$ must show its the period.

Note: $x_0 \in A_n$, $f(x_0) \in A_{n-1} \subseteq I$ and $f^i(x_0) \in J$ for $i = 2, 3, \dots, n$. Suppose $f^i(x_0) = x_0$ for some $i < n$.

Then, $f(x_0) = f^{i+1}(x_0) = b$ only overlap in both intervals
 $\downarrow$ $\downarrow$
 $\in I$ $\in J$

$\Rightarrow f(x_0) = b$, $f^2(x_0) = c$, $f^3(x_0) = a \Rightarrow a \in J$ but that's not true. $a \in I$. Hence x_0 has period n .

Further, $f(J) \supseteq J$ and so f has a fixed point (Prop 1) in J .

Finally, $f(I) \supseteq J \Rightarrow J = f(I')$

$f(J) \supseteq I' \Rightarrow f(J) = I'$

$\Rightarrow f^2(J) = f(I') = J \supseteq J' \Rightarrow \exists x_0 \in J'$ s.t. $f^2(x_0) = x_0$. by prop 1 If $f(x_0) = x_0$, $x_0 \in J'$ and $f(x_0) \in I' \Rightarrow x_0 = b$ but $f(b) \neq b$ b/c $f(b) = c$. Thus a contradiction. Hence x_0 has period 2

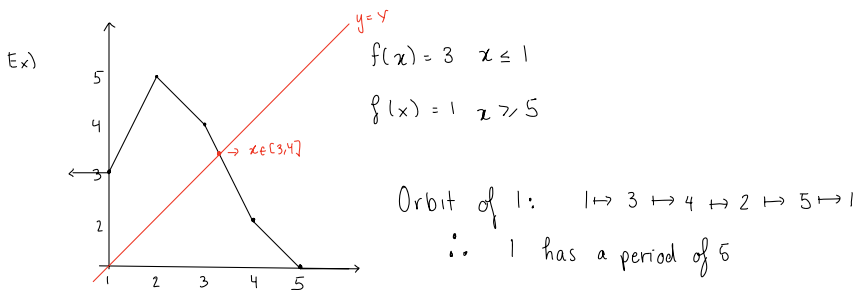
HOMEWORK: $a < c < b$

Lec 16 - Feb 12th 2023

Theorem: Period Three

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. If f has a point with period 3, then f has point with n for all $n \in \mathbb{N}$.

Proof is basically repeating applications of the Intermediate Value Theorem. 3 is important because you can create 2 intervals with the 3 numbers



Claim: f has no point with period 3. Suppose f has a point with period 3. Note $1 \leq x \leq 5$

$\rightarrow$ since we are assuming x is a 3-periodic point

$$\begin{aligned} \textcircled{1} \quad x \in [1, 2] & \quad x \in [1, 2] \cap f^3([1, 2]) \\ & = [1, 2] \cap [2, 5] \\ & \Rightarrow x = 2 \quad \text{but contradiction b/c 2 has period 5.} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad x \in [2, 3] & \Rightarrow x \in [2, 3] \cap f^3([2, 3]) \\ & = [2, 3] \cap [3, 5] \Rightarrow x = 3 \quad \text{but again contradiction} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad x \in [4, 5] & \Rightarrow x \in [4, 5] \cap f^3([4, 5]) \\ & = [4, 5] \cap [1, 4] \\ & \Rightarrow x = 4 \quad \text{but contradiction} \end{aligned}$$

$$\begin{aligned} \textcircled{4} \quad x \in [3, 4] & \Rightarrow x \in [3, 4] \cap f^3([3, 4]) \\ & = [3, 4] \cap [1, 5] \end{aligned}$$

$f: [3, 4] \rightarrow [2, 4]$ strictly decreasing

$f: [2, 4] \rightarrow [2, 5]$ strictly decreasing

$f: [2, 5] \rightarrow [1, 5]$ strictly decreasing.

$\therefore f^3(x)$ is strictly decreasing. $\rightarrow$ Since strictly decreasing $f^3(x)$ only crosses the line $y=x$ once

$\therefore f^3(x)$ has a unique fixed point in $[3, 4]$ but it is just the fixed point of f in $[3, 4]$

$\therefore f$ has no point with period three.

Sometimes a function has period n but not period k .

$$\text{Ex)} \quad f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \leq x \leq 1 \\ -1 & x > 1 \end{cases}$$

For $x < -1$ not periodic

$-1 \leq x \leq 1, x \neq 0$ period 2

$x = 0$ period 1

$x > 1$ not periodic

Sarkovskii Ordering

$$3 < 5 < 7 < 9 < 11 \dots$$

$\wedge$

$$2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < 2 \cdot 9 < 2 \cdot 11$$

$\wedge$

$$2^2 \cdot 3 < 2^2 \cdot 5 < 2^2 \cdot 7 < 2^2 \cdot 9 < 2^2 \cdot 11$$

$\vdots$

$\wedge$

$$\dots 2^n < 2^{n+1} < \dots < 2^2 < 2 < 1$$

This has created a total ordering of the natural numbers where 3 is the smallest, 2 is the second largest and 1 is the largest

$$\text{Ex} \quad 2 \cdot 13 < 2^2 \cdot 5$$

$$2^{10} \cdot 3 < 2^5 \quad \forall n \in \mathbb{N}, n \leq 1.$$

Theorem: Sarkovski

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose $n, m \in \mathbb{N}$ with $n < m$ (in Sarkovski ordering). If f has a point with period n , then it has a point with period m

Lec 17 - Feb 14 2024

"I'll switch up the polynomial, trig / exponential on the mid term..."

Chapter 9 - Fractals

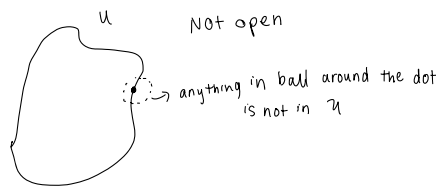
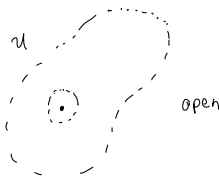
Def'n / Notation

① $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

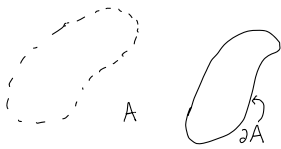
② $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ is a metric on $\mathbb{R}^n$

③ $\vec{x} \in \mathbb{R}^n$, $\epsilon > 0$. The open ball of radius epsilon centered at $\vec{x}$: $B_\epsilon(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \|\vec{x} - \vec{y}\| < \epsilon\}$

④ We say $U \subseteq \mathbb{R}^n$ is open if and only if $\forall \vec{x} \in U$, $\exists \epsilon > 0$, $B_\epsilon(\vec{x}) \subseteq U$



⑤ $A \subseteq \mathbb{R}^n$, $\partial(A)$ = boundary of A.



Precisely, the closure take away the interior.

Def'n We say $S \subseteq \mathbb{R}^n$ has topological Dimension 0 iff $\forall \vec{x} \in S$, $\exists$ arbitrarily small open $\vec{x} \in U$ s.t.

$$\partial(U) \cap S \neq \emptyset$$

Ex) $\overset{\text{call this } X}{\text{isolated points}} \rightarrow \text{this has dimension 0} \quad \dim_t X = 0$

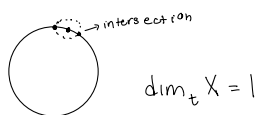
Ex) $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \quad \dim_t X = 0$ } non-isolated points

Because of the Hausdorff property of $\mathbb{R}$, doesn't this imply any Hausdorff $\dim_t(X) = 0$? ~~no~~ LOL

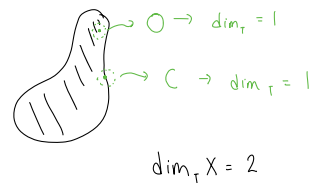
Def'n: $S \subseteq \mathbb{R}^n$, we say S has topological dimension $k \in \mathbb{N}$ iff $\forall \vec{x} \in S$ there exists arbitrarily small $\vec{x} \in U$ s.t. $\partial(U) \cap S$ has topological dimension $k-1$, where k is minimal with this property.

Ex) X $\dim_t = 0 \Rightarrow \dim_t(X) = 1$

Ex



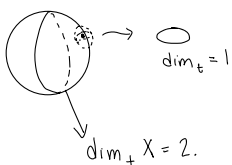
Ex)



$\rightarrow \dim_T = 1$ from previous example.

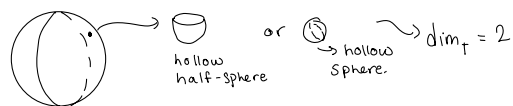
This is a recursive definition

Ex) X is a non-filled sphere



even though 3-D, Topological dim is 2

Ex) X is a filled-sphere



$$\dim_T X = 3.$$

$\Rightarrow$ filled sphere is bigger than a hollow sphere

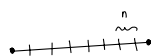
Defn: $S \subseteq \mathbb{R}^n$. we say S is Self-Similar if S may be divided into k Congruent Subsets each of which may be magnified by a fixed M to yield S itself

we then define the fractal dimension of S by

$$\dim_f S = \frac{\ln(k)}{\ln(M)}$$

Defn: A fractal is a self-similar $S \subseteq \mathbb{R}^n$ s.t. $\dim_f S > \dim_t S$.

Ex

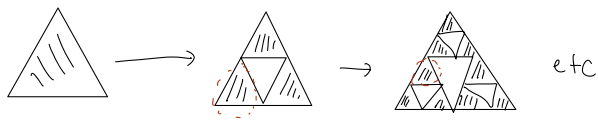


$$\dim_t X = 1$$

$$\dim_f X = \frac{\ln(n)}{\ln(n)} \rightarrow n \text{ partition}$$

$\rightarrow n$ is the multiplier to get back the line

Ex) X be Sierpinski Triangle



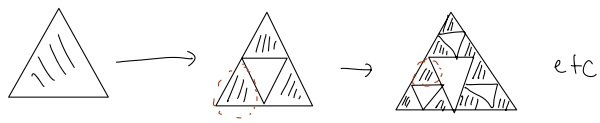
$$\dim_t X = 1$$

$$\dim_f X = \frac{\ln(3)}{\ln(2)} = \frac{\ln(9)}{\ln(4)} \quad \ln \text{ helps so that no matter what iteration, it gives same \#}$$
$$\approx 1.585 \dots$$

Distance between fractal dimension and Topological Dimension gives a measure of how crazy the fractal is - Sierpinski Triangle pretty tame.

Lec 18 - Feb 16th 2024

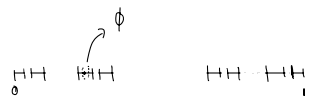
Ex) X be Sierpinski Triangle



$\dim_t X = 1$

$\dim_f X = \frac{\ln(3)}{\ln(2)} = \log_2 3$ → change of basis formula

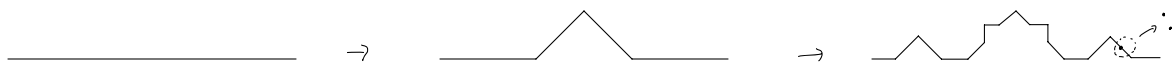
Ex) Cantor Set. K



For every point in the Cantor set, there are arbitrarily small amounts of points around that point. Hence $\dim_t K = 0$. Minimality comes into play here. b/c you could find a point near a boundary

$\dim_f K = \frac{\ln(2)}{\ln(3)}$ → pieces → magnification. $= \log_3(2) = 0.63$

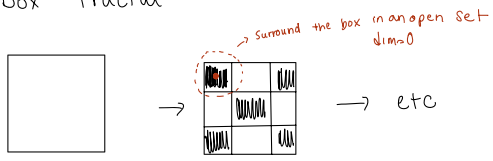
Ex) Koch Curve



This is a fractal. The Koch Snowflake is not totally a fractal according to our definition.

$\dim_t X = 1$, $\dim_f X = \frac{\ln(4)}{\ln(3)} = \log_3 4$

Ex) Box Fractal

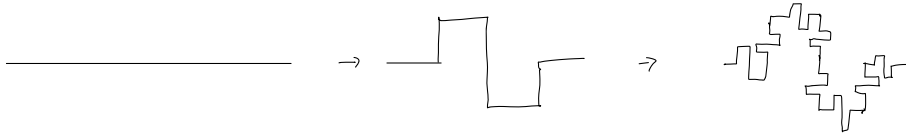


→ This is path connected

$\dim_t X = 1$

$\dim_f X = \frac{\ln(5)}{\ln(3)} = \log_3(5)$

Ex) Minkowski's Sausage



$$\dim_t X = 1, \quad \dim_f X = \frac{\ln(8)}{\ln(4)} = \log_4(8) = \frac{3}{2}$$

The Chaos game: [Shodor.org/Interactive/activities/The Chaos Game](http://Shodor.org/Interactive/activities/The%20Chaos%20Game).

↳ Can create fractals

↳ fractals can come from iterated function system

Google CGR DNA.

Lec 19 - Feb 26th 2024

Recall: Chaos Game.

① Start with the vertices of an equilateral triangle



② Pick $p \in \mathbb{R}^2$

③ Randomly Select a vertex v_i

④ Replace p with the mid point (p, v_i)

⑤ iterate.

Question: where does the orbit of p end up? Answer: Sierpinski triangle.

Goal: Formalize this.

Iterated Function Systems

Fix $p_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$ and fix $0 < \beta < 1$ (contraction factor)

Consider $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ ie $F(p) = \beta(p - p_0) + p_0$

Note:

① $F(p_0) = p_0 \rightarrow$ fixed point

② $\|F(p) - F(p_0)\| = \|\beta(p - p_0)\| = \beta \|p - p_0\|$

③ $\|F^n(p) - p_0\| = \beta^n \|p - p_0\| \rightarrow 0 \quad \therefore F^n(p) \rightarrow p_0$

Def'n: $0 < \beta < 1$, $p_1, p_2, \dots, p_n \in \mathbb{R}^2$

$F_i(p) = \beta(p - p_i) + p_i$. We call $\{F_1, F_2, \dots, F_n\}$ an iterated function System. Fix $p_0 \in \mathbb{R}^2$. Randomly select F_i . Let $q_1 = F_i(q_0)$ and Continuing...

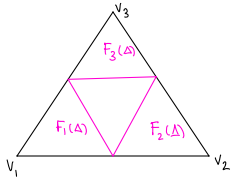
(iterated function system)

The set of points in which the orbit $q_1, q_2, q_3 \dots$ must eventually end up in, is called the attractor for the IFS

Ex) Chaos Game $P_1 = V_1$, $P_2 = V_2$, $P_3 = V_3$, $\beta = \frac{1}{2}$

$$F_i(p) = \frac{1}{2}(p - p_i) + p_i = \frac{1}{2}(p + p_i) = \text{midpoint}(p, p_i)$$

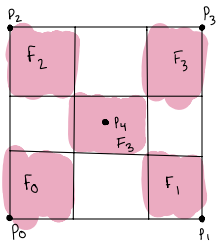
$\{F_1, F_2, F_3\}$ IFS and the attractor = Sierpinski Triangle.



Here we have
equal probability

"The orbits are dense in
the attractor set"

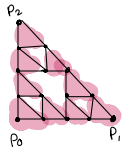
Ex) $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $p_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $p_4 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\beta = \frac{1}{3}$



Box fractal = Attractor.

Think where one iteration of functions go to

Ex $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\beta = \frac{1}{2}$



Attractor = Right Sierpinski triangle.

Lec 20 - Feb 29th 2028

Ex $P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\beta = \frac{1}{3}$

The IFS is:

$$F_0(\vec{x}) = \frac{1}{3} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x \\ \frac{1}{3}y \end{bmatrix}$$

$$F_1(\vec{x}) = \frac{1}{3} \begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{bmatrix}$$

$q_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$. Say the orbit of $\{F_0, F_1\}$ is $q_0, q_1, q_2, \dots, q_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ with random selections $s_1, s_2, s_3, \dots, s_i \in \{0, 1\}$

Note: $y_n = \frac{1}{3^n} y_0 \rightarrow 0$

$$x_1 = \frac{1}{3}x_0 + \frac{2s_1}{3}$$

$$x_2 = \frac{1}{3^2}x_0 + \frac{2s_1}{3^2} + \frac{2s_2}{3}$$

$$x_3 = \frac{1}{3^3}x_0 + \frac{2s_1}{3^3} + \frac{2s_2}{3^2} + \frac{2s_3}{3}$$

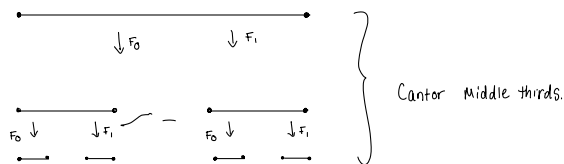
$$\vdots$$

$$x_n = \frac{1}{3^n}x_0 + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \frac{2s_3}{3^{n-2}} + \dots + \frac{2s_n}{3}$$

$\therefore x_n$ gets arbitrarily close to points of the form $\sum_{i=1}^{\infty} \frac{t_i}{3^i}$ where $t_i \in \{0, 2\}$.

Therefore the attractor of the IFS is $\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \text{Cantor Set} \}$

graphically:



(Generalized) Iterated Function Systems

Defn: An affine transformation is a transformation $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F(\vec{x}) = A(\vec{x}) + \vec{b}$ where $A \in M_n(\mathbb{R})$, $\vec{b} \in \mathbb{R}^n$

idea is that its basically a linear transformation b/c it doesn't need to pass through the origin

Moreover, we call F a linear contraction if $\exists 0 < \lambda < 1$ such that $\|F(\vec{x}) - F(\vec{y})\| < \lambda \|\vec{x} - \vec{y}\| \rightarrow$ There is a connection to the eigenvalues?

Ex $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ and $0 < \theta < 1$

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad F(\vec{x}) = \beta A \vec{x} + \vec{b}$$

$\rightarrow$ Scales by β

$\rightarrow$ rotates counter-clockwise by θ

$\rightarrow$ translates by $\vec{b}$

only 2 out of the 3 rigid motions
3rd in reflection

Def'n: we say $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.

Notation: $K_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact} \}$

Def'n: Let $F_1, F_2, \dots, F_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be linear contractions. We call $F: K_n \rightarrow K_n$ $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ an iterated function system

↓
not well defined in this defn but it is compact & Bounded on both sides

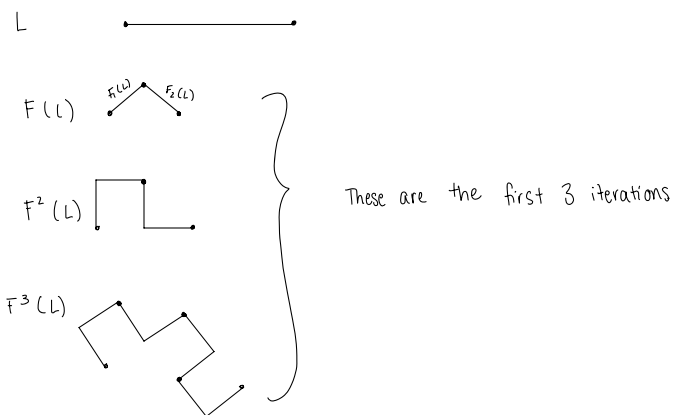
We will:

- ① Equip K_n with a metric ↗ v. similar to Banach contraction
- ② Show F has a unique fixed point A^* and $\forall A \in K_n, F^n(A) \rightarrow A^*$
- ③ we call A^* the attractor of F .

Ex) $F_1(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \vec{x} \rightarrow \text{this rotates by } \frac{\pi}{4}$
↓
squashes

$F_2(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \text{translates 1 step}$
↑

L = line segment from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Dragon Curve: Video by mathematical visual proofs

Lec 21 - Mar 4th 2024

2 speakers lined up for end of semester, trying to get as many speakers as he can
assignment due Mar 12th. May have class on April 8th

Recall:

$$F_1, F_2 \dots F_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

each $F_i(\vec{x}) = A_i \vec{x} + \vec{b}_i$ and $\|F_i(\vec{x}) - F_i(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$ where $\lambda \in (0,1)$ each F_i will have there own λ_i but here
 $\lambda = \max_i (\lambda_i)$. These are Linear Contractions.

Consider $\mathcal{K}_n = \{A \subseteq \mathbb{R}^n : A \text{ compact}\}$ remember compact = closed and bounded.

Fact: $\forall A \in \mathcal{K}_n \Rightarrow F_i(A) \in \mathcal{K}_n$ means we can iterate.

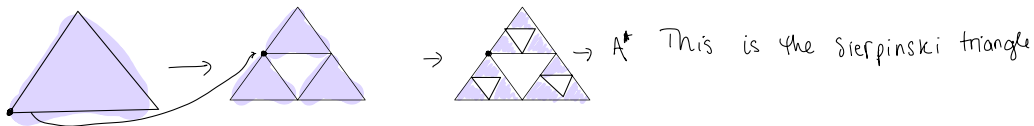
To do: Equip $\mathcal{K}_n$ with a metric. We will consider $F: \mathcal{K}_n \rightarrow \mathcal{K}_n$, $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ F gives you all of the images at once

To do: We show that F has a unique fixed point $A^* \in \mathcal{K}_n$ and $\forall A \in \mathcal{K}_n$, $F(A) \rightarrow A^*$ Like Banach contraction theorem, but we need metric

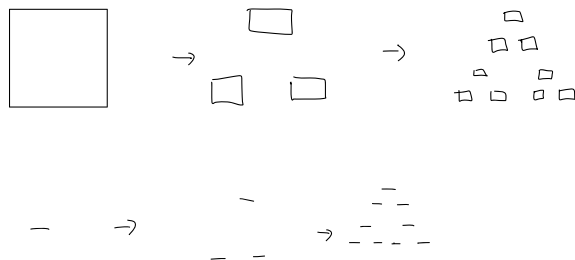
We call F on (generalized) iterated function system. We call A^* its attractor.

Ex) $F_1 = \frac{1}{2} \vec{x}$ ↗ $\frac{1}{2}$ Identity matrix
 $F_2 = \frac{1}{2} \vec{x} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$
 $F_3 = \frac{1}{2} \vec{x} + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$

Let A be the filled triangle with vertices $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$



OR.

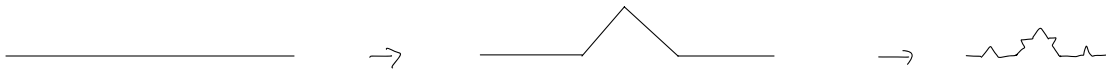


No matter what compact set you start out with, you converge to same thing

Ex) $F_1(\vec{x}) = \frac{1}{3} \vec{x}$
 $F_2(\vec{x}) = \frac{1}{3} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$ ↗ $\theta = \pi/3$
 $F_3(\vec{x}) = \frac{1}{3} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix}$ ↗ special θ for rotation matrix $\theta = -\pi/3$

$$F_4(\vec{x}) = \frac{1}{3} \vec{x} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

and $L =$ line segment from $(0,0)$ to $(1,0)$



The attractor
is the
Koch Curve

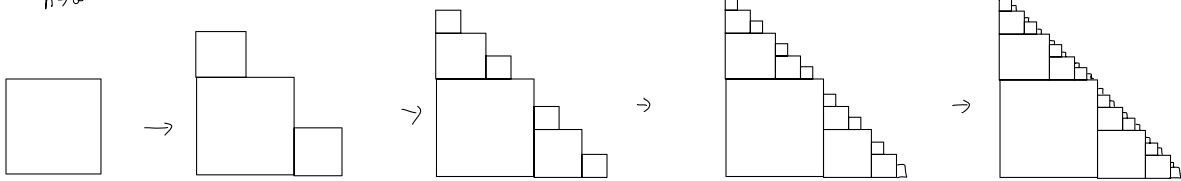
Ex) $A = [0,1] \times [0,1]$ the filled square

$$F_1(\vec{x}) = \vec{x}$$

$$F_2(\vec{x}) = \frac{1}{2} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_3(\vec{x}) = \frac{1}{2} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Remark: Not quite an IFS b/c F_1 is not a contraction. $\Rightarrow$ no longer always converge to an attractor. Will depend on I.C.
i.e. $\lim_{n \rightarrow \infty} f^n(A)$ will depend on A . But turns out for us limit does exist



not congruent shapes
but still very similar

Defn: [Hausdorff Metric]

$\vec{v} \in \mathbb{R}^n$, $A, B \in \mathcal{R}_n$

$$d(\vec{v}, B) = \min \{ \|\vec{v} - \vec{b}\| : \vec{b} \in B \} \quad \text{implies a min. Extreme Value theorem.}$$

$$d(A, B) = \max \{ d(a, B) : a \in A \} \rightarrow \text{The biggest shortest distance}$$

$$D(A, B) = \max \{ d(A, B), d(B, A) \}$$

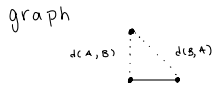
Lec 22 - Mar 6th 2024

Recall: The Hausdorff metric (D), formulation is in last lecture

↳ The way little d is not symmetric that the motivation for D b/c the max in it solves the symmetry issue

Fact: D is a metric on $\mathcal{K}_n$

Ex) $A = \{(1,1)\}$, $B = \{(x,0) : 0 \leq x \leq 1\}$



$$d(A,B) = 1$$

$$d(B,A) = \sqrt{2} \quad (a^2 + b^2 = c^2)$$

Q is class: Would min work? Probably not b/c triangle inequality probably fails

The reason we have a max in $d(A,B)$ is b/c we want to only have 2 set be the same is that if they are the same set

Finally, $D(A,B) = \max\{1, \sqrt{2}\} = \sqrt{2}$

Lemma 1: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear contraction such that $\|f(\vec{x}) - f(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$, $\lambda \in (0,1)$

For $A, B \in \mathcal{K}_n$ then $D(f(A), f(B)) \leq \lambda D(A,B)$

Proof: First,
$$\begin{aligned} d(f(A), f(B)) &= \min_{b \in B} \|f(a) - f(b)\| \\ &\leq \min_{b \in B} \lambda \|a - b\| = \lambda \min_{b \in B} \|a - b\| \\ &= \lambda d(a, B) \end{aligned}$$

and so
$$\begin{aligned} d(f(A), f(B)) &= \max_{a \in A} d(f(a), f(B)) \\ &\leq \lambda \max_{a \in A} d(a, B) = \lambda d(A, B) \\ &\leq \lambda D(A, B) \end{aligned}$$

comes from
defn & properties of min & max

$$\therefore d(f(A), f(B)) \leq \lambda D(A, B)$$

Similarly $d(f(B), f(A)) \leq \lambda D(A, B) \rightarrow$ convince yourself of this

$$\therefore D(f(A), f(B)) \leq \lambda D(A, B)$$

Lemma 2: For $A_1, A_2, B_1, B_2 \in \mathcal{K}_n$; $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$

Proof: First,
$$d(A_1 \cup A_2, B_1 \cup B_2) = \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2)$$

$$= \max \left\{ \max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2) \right\}$$

→ ensure biggest distance

$$\leq \max \left\{ \max_{a \in A_1} d(a, B_1), \max_{a \in A_2} d(a, B_2) \right\} \quad \text{By the min in the definition of } d(\vec{v}, B)$$

$$= \max \{d(A_1, B_1), d(A_2, B_2)\}$$

$$\leq \max \{D(A_1, B_1), D(A_2, B_2)\}$$

$$\therefore d(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{D(A_1, B_1), D(A_2, B_2)\}$$

Similarly $d(B_1 \cup B_2, A_1 \cup A_2) \leq \max \{D(A_1, B_2), D(A_2, B_1)\}$

$$\therefore D(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{D(A_1, B_1), D(A_2, B_2)\}$$

Lemma 3: Let $F_1, \dots, F_k$ be linear contractions with contraction factor $\lambda \in (0, 1)$

Consider $F: K_n \rightarrow K_n$, $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$. Then $D(F(A), F(B)) \leq \lambda D(A, B)$

Proof: We have, $D(F(A), F(B)) \leq \max_{i=1, \dots, k} D(F_i(A) \cup F_i(B))$ By the inductive generalization of lemma 2

$$\leq \max_{i=1, \dots, k} \lambda D(A, B) \quad \text{By lemma 1}$$

$$= \lambda D(A, B) \quad \text{b/c no more } i\text{-dependency. This is technically a corollary}$$

Q from class: Can be infinitely many? probs not b/c we may lose compactness.

Defn: (X, d) metric space.

① We say $(x_n) \subseteq X$ is Cauchy iff $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$.

② We say X is complete iff every Cauchy sequence $(x_n) \subseteq X$ converges to some $x \in X$.

Fact: (K_n, D) is complete! yay for us

↳ useful for us

Lec 23 - Mar 8 2024

Guest speakers! right now 4 but hoping to get 5. This will take last 2 weeks of guest speaker. Speaker content will be on final. 50% of final MC and the content may show up there.

Correction: $K_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact}, A \neq \emptyset \}$

Recall: $F_i: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad 1 \leq i \leq k$ are linear contractions

$$\|F_i(\vec{x}) - F_i(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$$

$$F: K_n \rightarrow K_n$$

$$F(A) = F_1(A) \cup \dots \cup F_k(A) \quad \text{then}$$

$$D(F(A), F(B)) \leq \lambda D(A, B)$$

B/c the function

contracts so does the

distance

Theorem

Let $F_1, \dots, F_k$ be linear contraction with contraction factor $\lambda \in (0, 1)$. Let $F: K_n \rightarrow K_n$ be the corresponding IFS. Then:

① F has a unique fixed point A^*

② For all $A^* \in K_n$, $F^m(A) \rightarrow A^*$.

We call A^* the attractor of F .

Can there be IFS that are not contractions such that there are multiple fixed points? like a kaleidoscope

→ one of our example of square stair case is an example of this

Proof:

Fix $A \in K_n$. Consider the Orbit $F^m(A)$, then,

$$\begin{aligned} D(F^{m+1}(A), F^m(A)) &= D(F^m(F(A)), F^m(A)) \\ &\leq \lambda^m D(F(A), A) \end{aligned}$$

call this ϵ_m

Note: $\sum \epsilon_m < \infty$, $|\lambda| < 1$. Therefore $(F^m(A)) \subseteq K_n$ is strongly Cauchy. technically we didn't define it in general space. but just change R defn. from $\| \cdot \|$ to D .

$\Rightarrow (F^m(A))$ is Cauchy $\Rightarrow F^m(A) \rightarrow A^* \in K_n$ since K_n is complete

Since F is continuous $F^{m+1}(A) \rightarrow F(A^*)$ Hence, $F(A^*) = A^*$

Suppose A^*, B^* are fixed points for F . then

$$\begin{aligned} D(A^*, B^*) &= D(F(A^*), F(B^*)) \leq \lambda D(A^*, B^*) \quad \lambda \in (0, 1) \\ \Rightarrow D(A^*, B^*) &= 0 \quad \Rightarrow A^* = B^* \end{aligned}$$

Chapter 10 - Complex functions

assume polar form. and complex algebra

these are modulus not absolute value

Defn $f(z)$, $\mathbb{C}$ -valued, $z \in \mathbb{C}$

① For $z_0 \in \mathbb{C}$, we say $\lim_{z \rightarrow z_0} f(z) = L \in \mathbb{C}$ iff $\forall \epsilon > 0, \exists \delta > 0, 0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon$

② We define the derivative of $f(z)$ at z_0 by $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$, provided the limit exists.

Defn: Let $a \in \mathbb{C}$ be a fixed point of $f(z)$.

① We say a is attracting $\Leftrightarrow |f'(a)| < 1$

① We say a is repelling $\Leftrightarrow |f'(a)| > 1$

① We say a is neutral $\Leftrightarrow |f'(a)| = 1$

Remark We obtain the $\mathbb{C}$ -analogue of the attracting / repelling fixed point theorems.

Ex) $f(z) = z^2 + z + 1$

Fixed points: $z^2 + z + 1 = z \Leftrightarrow z^2 + 1 = 0 \Leftrightarrow z = \pm i$

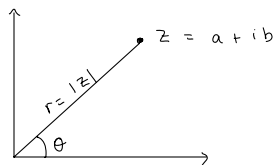
$f'(z) = 2z + 1$

$|f'(i)| = |2i + 1| = \sqrt{4+1} = \sqrt{5} > 1$

$|f'(-i)| = |-2i + 1| = \sqrt{5} > 1$

$\Rightarrow$ repelling

Recall



and $\theta = [0, 2\pi)$

$\therefore z = r(\cos \theta + i \sin \theta) = re^{i\theta}$

Fact: $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta+\phi)}$

$(re^{i\theta})^n = r^n e^{in\theta}$

Recall: $(e^{\frac{2\pi i}{n}})^n = e^{2\pi i} = 1$ This gets of periodic points
 n^{th} roots of unity.

Lee 24 - Mar 11th 2023

Assignment 2 due tomorrow.
 hint:
 2: not chaotic, focus on transitivity

Ex) $z = e^{2\pi i/3}$
 $= \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$
 $= -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

$f(w) = w^2$ *very easy in this form*
 $f(z) = (e^{2\pi i/3})^2 = e^{4\pi i/3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$ *from cos & sin*
 $f^2(z) = e^{8\pi i/3} = e^{2\pi i/3} = z$ *reducing by 2π ?*

period 2 *derivative formula from before works in $\mathbb{C}$*

$|(f^2)'(z)| = |f'(z) \cdot f'(f(z))|$
 $= |(-1 + i\sqrt{3})(-1 - i\sqrt{3})|$
 $= 2 \cdot 2 = 4 > 1$

$\therefore z$ is a repelling periodic point for $f(w)$

Chapter 11 Julia Sets

Notation: For $c \in \mathbb{C}$, $Q_c(z) = z^2 + c$

orbit is bounded.

Def'n: Let $c \in \mathbb{C}$. Then the filled Julia Set for c is $K_c = \{z \in \mathbb{C} : (Q_c^n(z)) \text{ is bounded}\}$
 $= \{z \in \mathbb{C} : \exists M > 0, \forall n \in \mathbb{N}, |Q_c^n(z)| \leq M\}$ *Complex analogue of Λ*

Remark: This is the complex analogue of Λ for $G_c(x) = x^2 + c$ where $c \in \mathbb{R}$ and $c < -2$.

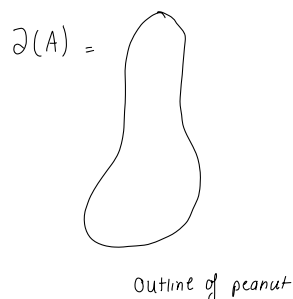
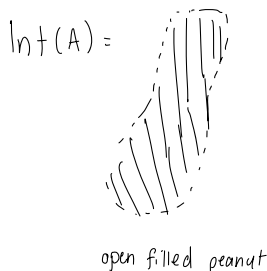
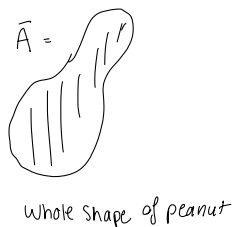
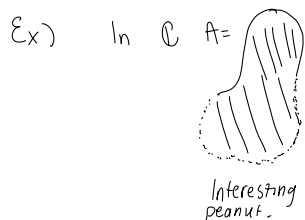
Def'n (X, d) , $A \subseteq X$

includes point on boundary which may not be in the set.

① the closure of A is $\bar{A} = \{x \in X, \exists (a_n) \subseteq A \text{ s.t. } a_n \rightarrow x\}$

② the interior of A is $\text{int}(A) = \{x \in A : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq A\}$

③ the boundary of A is $\partial(A) = \bar{A} \setminus \text{int}(A)$



Remark: A is closed iff $A = \bar{A}$

Assignment 4: Will show K_c is closed $\rightarrow$ means J_c won't contain any new points from the filled Julia Set

Def'n: Let $C \in \mathbb{C}$. The Julia Set for C is $J_c = \partial(K_c)$

Remark: $J_c = \partial(K_c) = \bar{K}_c \setminus \text{Int}(K_c) = K_c \setminus \text{Int}(K_c)$

Ex) $C=0$, $Q_0(z) = z^2$

$z = r e^{i\theta}$ $\rightarrow$ r is modulus of z , $\cos^2 + \sin^2 = 1$

$$|Q_0(z)| = |r^2 e^{2i\theta}| = r^2$$

$$|Q_0^2(z)| = |r^4 e^{4i\theta}| = r^4$$

$$|Q_0^3(z)| = |r^8 e^{8i\theta}| = r^8$$

$\vdots$

$$|Q_0^n(z)| = r^{2^n}$$

if $r \in [0, 1] \Rightarrow Q_0^n(z)$ bounded

$$K_0 = \{z \in \mathbb{C} : |z| \leq 1\}.$$

$$J_0 = \{z \in \mathbb{C} : |z| = 1\} \rightarrow \text{unit circle}$$

Ex) $C = -2$

$$Q_{-2}(z) = z^2 - 2$$

$$\textcircled{1} R = \{z \in \mathbb{C} : |z| > 1\}$$

$$H: R \rightarrow \mathbb{C}$$

$$H(z) = z + \frac{1}{z}$$

} not clear at first.

H is a translation function, aim to bridge information b/w Q_0 and Q_{-2}

$$\textcircled{2} \text{ Claim } H \text{ is Injective: } H(z) = H(w) \Rightarrow z + \frac{1}{z} = w + \frac{1}{w}$$

$$\Rightarrow z w = z^2 + 1 - \frac{z}{w} = \overset{\times z}{w^2 + 1} - \frac{w}{z}$$

$$\Rightarrow w^2 - z^2 = \frac{w}{z} - \frac{z}{w} = \frac{w^2 - z^2}{zw} \rightarrow \text{only if } zw = 1 \text{ or } w^2 - z^2$$

Since $w, z \in R$, we have $w^2 - z^2 = 0 \Rightarrow w = \pm z$.

$$\text{Since } H(w) = H(z) \Rightarrow w = z$$

But $zw \neq 1$
b/c $|z|$ and $|w| > 1$

$$\textcircled{3} H: R \rightarrow \mathbb{C} \setminus [-2, 2] \text{ is surjective}$$

$$H(z) = w \Leftrightarrow z + \frac{1}{z} = w$$

$$\Rightarrow z^2 - wz + 1 = 0$$

$$\Rightarrow z_{\pm} = \frac{1}{2}(w \pm \sqrt{w^2 - 4})$$

Note: $z_+ z_- = 1$

if $z_+ \in \mathbb{R}$ or $z_- \in \mathbb{R}$ $H(z_+) = \omega$ or $H(z_-) = \omega$. Otherwise $|z_{\pm}| = 1$.

Say $z_{\pm} = e^{i\theta}$

$$H(z_{\pm}) = H(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in [-2, 2]$$

lec 25 - Mar 13th 2024

Recall: $K_c = \{z \in \mathbb{C} : Q_c(z) \text{ bounded}\}$

$$J_c = \partial(K_c)$$

$$K_0 = \{z : |z| \leq 1\}$$

$$J_0 = \{z : |z| = 1\}$$

Ex: $c = -2$ $Q_{-2}(z) = z^2 - 2$

① $R = \{z : |z| > 1\}$ $H: R \rightarrow \mathbb{C}$ $H(z) = z + \frac{1}{z}$

② H injective

③ $H: R \rightarrow \mathbb{C} \setminus [-2, 2]$ surjective

④ $H(Q_0(z)) = H(z^2) = z^2 + \frac{1}{z^2}$

$$Q_{-2}(H(z)) = \left(z^2 + \frac{1}{z^2}\right)^2 - 2 = z^4 + \frac{1}{z^4}$$

$\begin{matrix} z^2 + \frac{1}{z^2} + 2\left(\frac{z^2}{z^2}\right) - 2 \\ \uparrow \quad \quad \quad \uparrow \end{matrix}$

$$H(Q_0^{\circ n}(z)) = Q_{-2}^{\circ n}(H(z))$$

Compare:

$$S(Q_c^{\circ n}(x)) = \sigma^n(S(x))$$

→ H is playing the same role
→ said to be conjugate but we won't define it.

⑤ $|z_n| \rightarrow \infty$

$$|H(z_n)| = \left|z_n + \frac{1}{z_n}\right|$$

$$\geq |z_n| - \left|\frac{1}{z_n}\right| \rightarrow \infty \text{ Reverse triangle inequality}$$

→ How does a complex # go to infinity?

$$\Rightarrow |H(z_n)| \rightarrow \infty$$

⑥ $z \in \mathbb{C} \setminus [-2, 2]$ $z = H(w)$, $w \in \mathbb{R}$

$$|Q_{-2}^{\circ n}(z)| = |Q_{-2}^{\circ n}(H(w))|$$

$$= |H(Q_0^{\circ n}(w))| \rightarrow \infty$$

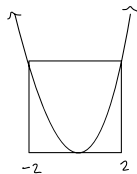
→ b/c $w \in \mathbb{R}$, $z > 1 \Rightarrow z \notin K_0 \Rightarrow Q_0^{\circ n}(z) \rightarrow \infty$

$$\therefore z \notin K_{-2}$$

⑦ Let $z \in [-2, 2]$

By graphical analysis, $z \in K_c$

→ bounded



Therefore $K_{-2} = [-2, 2]$

$$J_{-2} = [-2, 2] \rightarrow \text{this is the boundary in the complex plane}$$

These are the only 2 Julia Sets that aren't fractals

[Escape criteria]

Proposition If $|z| \geq |c| > 2$ then $|Q_c^n(z)| \rightarrow \infty$. In particular, $z \notin K_c$

proof: we have $|Q_c(z)| = |z^2 + c| \geq |z|^2 - |c| \xrightarrow{\text{Reverse triangle}} |z|^2 - |z| \xrightarrow{\text{from proposition}} |z|(|z| - 1)$

Say $|z| > 2 + \lambda$, $\lambda > 0$. $\Rightarrow |z| - 1 > 1 + \lambda$

Hence $|Q_c(z)| \geq |z|(1 + \lambda)$

$\Rightarrow |Q_c^n(z)| \geq |z|(1 + \lambda)^n \rightarrow$ how come $|z|$ doesn't get raised to the n

and $(1 + \lambda)^n \rightarrow \infty$ Hence $|Q_c^n(z)| \rightarrow \infty$

Corollary: $|c| > 2$ Then $|Q_c^n(c)| \rightarrow \infty$ so that $0 \notin K_c$

Why?

$$z = Q_c(0) = c$$

$$|z| = |c| > 2$$

$\Rightarrow |Q_c^n(z)| \rightarrow \infty$ By previous proposition

$\Rightarrow |Q_c^n(0)| \rightarrow \infty$

Corollary: $M = \max \{ |c|, 2 \}$ ^{some complex #}
If $|z| > M$ then $|Q_c^n(z)| \rightarrow \infty$. Thus, $K_c \subseteq \{z : |z| \leq M\}$ ^{hence bounded sets}

Why?

$$|Q_c^n(z)| \geq (1 + \lambda^n) |z| \rightarrow \infty$$

Cannot use escape criterion but use exact same proof b/c the assumption $|z| \geq |c| > 2$ in the Escape criterion is not met.

Hint we haven't showed K_c is closed but knowing $K_c \subseteq \{z : |z| \leq M\}$ helps in showing that

Corollary: if $\exists k$ s.t. $|Q_c^k(z)| > \max \{ |c|, 2 \}$ then $|Q_c^n(z)| \rightarrow \infty$ thus $z \notin K_c$

Algorithm:

- ① choose a large $N \in \mathbb{N}$
- ② if $|Q_c^i(z)| > \max \{ |c|, 2 \}$ for $i \leq N$ colour z white
- ③ if $|Q_c^i(z)| \leq \max \{ |c|, 2 \}$ for all $i \leq N$, colour z black

The black shaded region approximates K_c

Lec 26th - Mar 15th 2024

Recall: If $\exists k$ s.t. $|Q_k(z)| > \max\{1, 2\}$ then $|Q_k(z)| \rightarrow \infty$. In particular, $z \notin K_c$.

Ex) Is $i \in K_{2+i}$?

$$Q(z) = z^2 + 2 + i \quad c = 2 + i \quad |c| = \sqrt{5}$$

$$\text{Max} = \max\{\sqrt{5}, 2\} = \sqrt{5}$$

$$i \mapsto 1+i \mapsto 2+3i$$

image under $Q(z)$
 $|1+i| = \sqrt{2}$

$$|2+3i| = \sqrt{4+9} = \sqrt{13} > \sqrt{5}$$

$\therefore$ No! $i \notin K_{2+i}$

K_c tends to be sparse so it was highly likely $i \notin K_{2+i}$

Remark: $n \in \mathbb{Z}$, $n \neq 0$

$$\int_0^{2\pi} e^{int} dt = \int_0^{2\pi} \cos(nt) + i \sin(nt) dt$$

$$\stackrel{\text{by defn}}{=} \int_0^{2\pi} \cos(nt) dt + i \int_0^{2\pi} \sin(nt) dt$$

$$= \left[\frac{1}{n} \sin(nt) \right]_0^{2\pi} + i \left[-\frac{1}{n} \cos(nt) \right]_0^{2\pi}$$

$0-0=0$ $-1-(-1)=0$

= 0 $\rightarrow$ it being zero is useful and important to Fourier analysis

if $n=0$,

$$\int_0^{2\pi} e^{int} dt = \int_0^{2\pi} 1 dt = 2\pi.$$

Prop [Cauchy's Estimate] $\rightarrow$ not most general

Let $P(z) = \sum_{n=0}^d a_n z^n$ be a polynomial s.t. $|P(z)| \leq M$ for all $|z - z_0| \leq r$ then

$$|P'(z_0)| \leq \frac{M}{r}$$

Proof

① assume $z_0 = 0$

Assume $|P(z)| \leq M$ for $|z| \leq r$

$$\text{Then, } \frac{1}{2\pi} \int_0^{2\pi} \frac{P(re^{it})}{re^{it}} dt$$

$\rightarrow \div$ by re^{it} gives -1

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^d a_n r^{n-1} e^{i(n-1)t} dt$$

When $n-1=0$ the integral isn't zero

$$= a_1$$

$$= P'(0)$$

from integral properties of a absolute value

$$\text{and so } |P'(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(re^{it})}{re^{it}} \right| dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{r} dt = \frac{M}{r}$$

Case 2 General Case.

$$\text{Assume } |P(z)| < M \text{ for } |z - z_0| \leq r$$

$$\text{Let } w = z - z_0 \text{ so that } |P(w + z_0)| \leq M \text{ for all } |w| \leq r.$$

Liouville notation.

$$\text{By Case 1, } \left| \frac{d}{dw} P(w + z_0) \right|_{w=0} \leq \frac{M}{r}$$

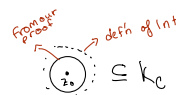
are absolute values

$$\Rightarrow \left| \frac{d}{dz} P(z) \right|_{z=z_0} \leq \frac{M}{r}$$

Theorem: If z_0 is a repelling periodic point for $Q_c(z)$, then $z_0 \in J_c$.

Proof: Assume z_0 is a repelling periodic point with period n . Suppose $z_0 \notin J_c$. Since z_0 is periodic, $z_0 \in K_c$.

Thus $z_0 \in \text{int}(K_c)$ and so $\exists r > 0$ s.t. $z \in K_c$ for all $|z - z_0| \leq r$.



Using escape criterion (contra positive)

For all z s.t. $|z - z_0| \leq r$ and all $k \in \mathbb{N}$, $|(Q_c^n)^k(z)| \leq M$, where $M = \max\{1, 2\}$.

By the Cauchy estimate.

$$\left| (Q_c^n)^k(z_0) \right| \leq \frac{M}{r} \text{ for all } k \in \mathbb{N}.$$

Havent used repelling yet.

$$\text{Say } |(Q_c^n)'(z_0)| = \lambda > 1$$

Blake called it

acute proof. it

may be on the final

$$\text{Finally, } |(Q_c^n)^k(z_0)| = |(Q_c^n)'(z_0)| \cdot |(Q_c^n)'(Q_c(z_0))| \cdot \dots \cdot |(Q_c^n)'(Q_c^{k-1}(z_0))|$$

$$= \lambda^k \rightarrow \infty \text{ which contradicts the Cauchy estimate}$$

Fact: $K \subset \mathbb{C}$ closed. If $z \in \text{int}(K)$ then $Q_c(z) \in \text{int}(Q_c(K))$

Proposition: If $Q_c(z) \in J_c$, then $z \in J_c$ the Julia set of the preimage.

Why? $Q_c(z) \in J_c \Rightarrow Q_c(z) \in K_c \Rightarrow z \in K_c$

$$Q_c(z) \notin \text{int}(K_c) \stackrel{\text{fact}}{\Rightarrow} z \notin \text{int}(K_c) \text{ note } Q_c(K_c) = K_c \Rightarrow z \in J_c$$

Lec 27 - Mar 18th 2024

Friday next week is holiday. Wednesday next week will start Guest speaker.

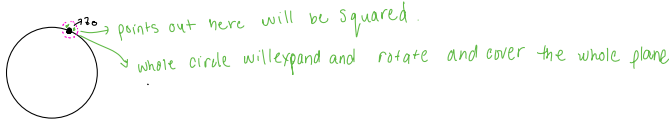
Recall: if $Q_c(z) \in J_c \Rightarrow z \in J_c$

Def'n: We say Q_c is super sensitive at $z_0 \in \mathbb{C}$ iff whenever $U \subseteq \mathbb{C}$ is open such that $z_0 \in U$ then $\mathbb{C} = \bigcup_{n=0}^{\infty} Q_c^n(U)$

functions explode they cover the complex numbers

Fact: $\forall z_0 \in J_c, Q_c$ is super sensitive at z_0

Ex. J_0 is the unit circle



Big Idea:

- ① pick $z \in \mathbb{C}$ and $z_0 \in J_c$
- ② For $\epsilon > 0$, $U = B_\epsilon(z_0)$
- ③ by Supersensitivity, $\exists k, \exists w \in U$ s.t. $Q_c^k(w) = z$
- ④ By looking at the backwards orbit of z . We can find a very close $w \in \mathbb{C}$ to J_c .

Why b/c $w \in U \Rightarrow$ close to Julia set.

More precisely:

- ① chose $z \in \mathbb{C}$
- ② Compute 10 000 terms in the backwards orbit, randomly selecting a pre-image at each step.
- ③ Plot all but the first 100 points from the backwards orbit. $\rightarrow$ b/c consider sufficient. for numerics we aren't talking about

b/c quadratic $\Rightarrow \pm$ roots $\Rightarrow$ pick one randomly

Path connected, for $c \in$ Mandelbrot set, the Julia set is path connected.

Different $c \in$ Mandelbrot set will have different periodic points

The Mandelbrot set is a dictionary for the Julia sets

Chapter 12 - The Mandelbrot Set

Def'n: for (X, d)

- ① Let $a, b \in X$. A path from a to b is a continuous function $\gamma: [0, 1] \rightarrow X$ s.t. $\gamma(0) = a$ and $\gamma(1) = b$



- ② We say $A \subseteq X$ is path connected iff $\forall a, b \in A \exists$ a path $\gamma: [0, 1] \rightarrow X$ from a to b s.t. $\gamma([0, 1]) \subseteq A$

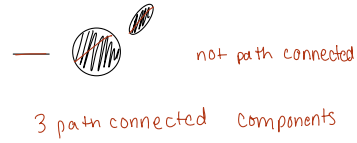
image



This is path connected



③ The maximal path connected subsets of A are called the path connected components of A .



④ We say A is totally disconnected iff the path connected components of A are exactly $\{a\}, a \in A$

Ex. Cantor Set.

not necessarily isolated

Lec 28 - Mar 20th 2024

Recall: $A \subseteq \mathbb{C}$

① A is path connected iff $\forall a, b \in A \exists$ a continuous $\gamma: [0, 1] \rightarrow \mathbb{C}$, $\gamma(0) = a$, $\gamma(1) = b$ $\gamma([0, 1]) \subseteq A$

② A is totally disconnected iff $\forall a \neq b$ there is no path in A from a to b

Ex. $A = \{z: |z| \leq 1\}$



Ex. A : not path connected but not totally disconnected

Ex. $K \subseteq \mathbb{R}$ Cantor Set is totally disconnected. ** remember that the Cantor set is not isolated.*

Theorem: For $Q_c(z) = z^2 + c$ either:

① $|Q_c^n(0)|$ is bounded ($0 \in K_c$) in which case K_c is path connected

or

② $|Q_c^n(0)| \rightarrow \infty$, $[0 \notin K_c]$, in which case K_c is totally disconnected.

Def'n: The Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C}: |Q_c^n(0)| \text{ is bounded}\}$
 $= \{c \in \mathbb{C}: |Q_c^n(0)| \not\rightarrow \infty\}$
 $= \{c \in \mathbb{C}: 0 \in K_c\}$
 $= \{c \in \mathbb{C}: K_c \text{ path connected}\}$

Recall: If $|c| > 2$, $|Q_c^n(0)| \rightarrow \infty \Rightarrow \mathcal{M} \subseteq \{z: |z| \leq 2\}$

Ex. $0, -2 \in \mathcal{M}$

$c = 0$ $c = -2$ — both are path connected and contain zero

Ex) $c = 2$ $Q_c(z) = z^2 + 2$

$0 \mapsto 2 \mapsto 6 \mapsto 38$

$|Q_c^n(0)| \rightarrow \infty$ $2 \notin \mathcal{M}$

Ex) $c = i$

$Q_c(z) = z^2 + i$

$0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i \mapsto i \dots$ Eventually periodic

$\therefore |Q_0^n(o)| \text{ bounded} \Rightarrow i \in \mathcal{M}$

Goal: Describe the shape of $\mathcal{M}$.



Lec 2a - Mar 22nd 2024

Assignment due next next tuesday. Guest speakers start Wednesday next week.
no class april 8th. Exam break down Monday, Theorem list on Monday

Recall: $\mathcal{M} = \{c \in \mathbb{C} : Q_c^n(0) \text{ is bounded}\}$
 $= \{c \in \mathbb{C} : 0 \in K_c\}$
 $= \{c \in \mathbb{C} : |Q_c^n(0)| \not\rightarrow \infty\}$
 $= \{c \in \mathbb{C} : K_c \text{ is path connected}\}$

Recall: $c \notin \mathcal{M}$ K_c is totally disconnected

Remark: $z \in \mathbb{D}$, $Q_c = z^2 + c$

$M = \max\{|c|, 2\}$. by the escape criterion

$$(1) \forall n, |Q_c^n(z)| \leq M$$

Something subtle, not all limits that are unbounded go to infinity like $0, 1, 0, 2, 0, 3, \dots$

$$(2) |Q_c^n(z)| \rightarrow \infty$$

Goal: Describe the general shape of $\mathcal{M}$:

Investigation:

(1) When does $Q_c(z)$ have an attracting fixed point?

Suppose $z \in \mathbb{C}$ is an attracting fixed point for Q_c . $\exists r > 0$, $x \in B_r(z) \Rightarrow Q_c^n(x) \rightarrow z$ Complex Analogue of a trapping fixed pt. theorem

$$\Rightarrow B_r(z) \subseteq K_c$$



$$\Rightarrow K_c \text{ path connected} \Rightarrow c \in \mathcal{M}$$

$$\text{Now, } z^2 + c = z, |2z| < 1$$

$$\Rightarrow c = z - z^2, |z| < 1/2$$

Let's explore the boundary: The most extreme value is when $|z| = 1/2$

$$z = \frac{1}{2} e^{i\theta}, c = z - z^2$$

$$c = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{2i\theta} \text{ where } \theta \in \mathbb{R} \text{ famous shape, cardioid}$$

dotted bc boundary is not included in attracting fixed point

$$\text{cardioid} \subseteq \mathcal{M}$$

② When does Q_c have an attracting 2-cycle?

If Q_c has an attracting periodic point z with period 2, we can similarly justify that $c \in \mathcal{M}$.

We know: $(z^2 + c)^2 + c = z$ simplified, try for yourself

$\Rightarrow z^4 + 2cz^2 - z + c^2 + c = 0$ the fixed points of Q_c are also roots of this polynomial.

Let p_1 & p_2 be the fixed points of Q_c

$\therefore (z - p_1)(z - p_2) = z^2 - z + c$ is a factor of the previous polynomial.

Thus, z is a root of $\frac{z^4 + 2cz^2 - z + c^2 + c}{z^2 - z + c}$ $\rightarrow$ this will isolate the roots that are 2-cycles

$= z^2 + z + c + 1$ $\rightarrow$ polynomial division try for yourself.

Let z_1 and z_2 be the roots of $z^2 + z + c + 1$. i.e. z_1, z_2 are the period 2 points for Q_c . For these to be attracting $|(Q_c^2)'(z)| < 1$.

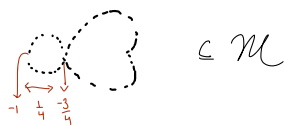
$$\Rightarrow |Q'(z_1) \cdot Q'(z_2)| < 1$$

$$\Rightarrow |2(z_1) \cdot 2(z_2)| < 1$$

$$\Rightarrow 4|z_1 z_2| < 1$$

$$\Rightarrow |z_1 z_2| < \frac{1}{4}$$

$\Rightarrow |c + 1| < \frac{1}{4}$ $\rightarrow$ Circle of radius $< \frac{1}{4}$ starting at -1 .



If you do this for n then you get Mandelbrot set. There are algorithms for the Mandelbrot set to visualize it. We don't touch that.

Proposition: $\mathcal{M} = \{c \in \mathbb{C} : |c| \leq 2\} \cap \{c : |c^2 + c| \leq 2\} \cap \{c : |(c^2 + c)^2 + c| \leq 2\} \cap \dots$

HOME WORK!

$\hookrightarrow$ should be algorithm for Computer

each of these are a preimage of a closed sets from -2 to 2

Proposition: $\mathcal{M}$ is closed.



Lec 30 - March 25th 2024

Last traditional lecture. Guest speakers start Wednesday

Chapter 13 - Polynomial Julia Sets

Theorem: [Polynomial Escape criterion]

Let $P(z) = a_n z^n + \dots + a_1 z + a_0$ be a complex polynomial $[\mathbb{C} \rightarrow \mathbb{C}]$ with $a_n \neq 0$ and $n \geq 2$. There exists $R > 0$ depending only on n and a_i s.t. $|P^k(z)| \rightarrow \infty$ for all $|z| \geq R$

Proof:

Fix $\lambda > 1$ and let $C = \sum_{i=0}^{n-1} |a_i|$. Consider $R = \max \left\{ 1, \frac{2C}{|a_n|}, \left(\frac{2\lambda}{|a_n|} \right)^{\frac{1}{n-1}} \right\}$. Assume $|z| \geq R$. Then,

$$\begin{aligned} |P(z)| &\geq |a_n z^n| - |a_{n-1} z^{n-1} + \dots + a_1 z + a_0| \\ &\geq |a_n z^n| - (|a_{n-1}| \cdot |z|^{n-1} + \dots + |a_1| \cdot |z| + |a_0|) \\ &\geq |a_n z^n| - [C |z|^{n-1}] \end{aligned}$$

Note: $|z| \geq \frac{2C}{a_n} \Rightarrow \frac{C}{|z|} \leq \frac{|a_n|}{2}$

$$= |z|^n \left(|a_n| - \frac{C}{|z|} \right)$$

$$\geq |z|^n \left(|a_n| - \frac{|a_n|}{2} \right)$$

$$= |z|^n \cdot \frac{1}{2} |a_n|$$

$$= |z| \cdot \frac{1}{2} |a_n| \cdot |z|^{n-1}$$

Note: $|z|^{n-1} \geq \frac{2\lambda}{|a_n|}$

$$\geq |z| \cdot \frac{1}{2} |a_n| \cdot \frac{2\lambda}{|a_n|}$$

$$= |z| \cdot \lambda$$

Thus $|P^k(z)| \geq \lambda^k |z| \rightarrow \infty$

Remark: Either

① $\forall k, |P^k(z)| < R$

② $|P^k(z)| \rightarrow \infty$

Defn: Let $P(z)$ be a complex polynomial with degree ≥ 2 .

① the filled Julia Set of $P(z)$: $K_P = \{z \in \mathbb{C} : P^k(z) \text{ is bounded}\}$

② The Julia Set of $P(z)$ is $J_P = \partial(K_P)$

$$\text{Ex } p(z) = z^3 + z - 0.15$$

$$\text{Ex } z^3 + z - 0.2$$

$$\text{Ex } z^3 + z - 0.25$$

$$\text{Ex } z^4 + z^3 + z - 0.25$$

Multibrot Sets. Look up.

are all Julia Sets Symmetric?